Self-dual Grassmannian and Representations of $\mathfrak{gl}_N$, $\mathfrak{sp}_{2r}$, and $\mathfrak{so}_{2r+1}$

Kang Lu (joint with E. Mukhin & A. Varchenko)
Indiana University–Purdue University Indianapolis
lukang@iupui.edu

IUPUI

Abstract

We define a $\mathfrak{gl}_N$-stratification of the Grassmannian of N planes $\mathrm{Gr}(N,d)$. The $\mathfrak{gl}_N$-stratification consists of strata $\Omega_{\boldsymbol{\Lambda}}$ labeled by unordered sets $\boldsymbol{\Lambda}=(\lambda^{(1)},\dots,\lambda^{(n)})$ of nonzero partitions with at most N parts, satisfying a condition depending on d, and such that $(\otimes_{i=1}^n V_{\lambda^{(i)}})^{\mathfrak{sl}_N}\neq 0$. Here $V_{\lambda^{(i)}}$ is the irreducible $\mathfrak{gl}_N$-module with highest weight $\lambda^{(i)}$. We show that the closure of a stratum $\Omega_{\boldsymbol{\Lambda}}$ is the union of the strata $\Omega_{\boldsymbol{\Xi}}$, $\boldsymbol{\Xi}=(\xi^{(1)},\dots,\xi^{(m)})$, such that there is a partition $\{I_1,\dots,I_m\}$ of $\{1,2,\dots,n\}$ with $\mathrm{Hom}_{\mathfrak{gl}_N}(V_{\xi^{(i)}},\otimes_{j\in I_i}V_{\lambda^{(j)}})\neq 0$ for $i=1,\dots,m$.

We introduce and study the new object: the self-dual Grassmannian $\mathrm{sGr}(N,d)\subset\mathrm{Gr}(N,d)$. Our main result is a similar $\mathfrak{g}_N$-stratification of the self-dual Grassmannian governed by representation theory of the Lie algebra $\mathfrak{g}_{2r+1}:=\mathfrak{sp}_{2r}$ if $N=2r+1$ and of the Lie algebra $\mathfrak{g}_{2r}:=\mathfrak{so}_{2r+1}$ if $N=2r$.

The stratifications are motivated by the Gaudin models associated with the corresponding Lie algebras. The proofs of main theorems are also based on the bijective correspondence between the common eigenvector of Bethe algebra (Gaudin algebra) and spaces of polynomials.

Schubert cells

Let $\mathbb{C}_d[x]$ be the space of polynomials in x with complex coefficients of degree less than d. We have $\dim\mathbb{C}_d[x]=d$. Let $\mathrm{Gr}(N,d)$ be the Grassmannian of all N-dimensional subspaces in $\mathbb{C}_d[x]$.

The Schubert cell decomposition and the closure of a Schubert cell associated to a complete flag $\mathcal{F}$ are given by

$$\mathrm{Gr}(N,d)=\bigsqcup_{\lambda,\ \lambda_1\leqslant d-N}\Omega_\lambda(\mathcal{F}),\qquad \overline{\Omega}_\lambda(\mathcal{F})=\bigsqcup_{\substack{\lambda\subseteq\mu,\\ \mu_1\leqslant d-N}}\Omega_\mu(\mathcal{F}).$$

Let $\mathcal{F}(\infty)$ be the complete flag given by $\mathcal{F}(\infty)=\{0\subset\mathbb{C}_1[x]\subset\mathbb{C}_2[x]\subset\cdots\subset\mathbb{C}_d[x]\}$. For $z\in\mathbb{C}$, consider the complete flag $\mathcal{F}(z)=\{0\subset(x-z)^{d-1}\mathbb{C}_1[x]\subset(x-z)^{d-2}\mathbb{C}_2[x]\subset\cdots\subset\mathbb{C}_d[x]\}$.

Let $\boldsymbol{\Lambda}=(\lambda^{(1)},\dots,\lambda^{(n)})$ be a sequence of partitions with at most N parts ($\mathfrak{gl}_N$-weights) and let $\boldsymbol{z}=(z_1,\dots,z_n)\in(\mathbb{P}^1)^{\times n}$ such that $z_i\neq z_j$ if $i\neq j$.

Assuming $\sum_{s=1}^n|\lambda^{(s)}|=N(d-N)$, denote by $\Omega_{\boldsymbol{\Lambda},\boldsymbol{z}}$ the intersection of the Schubert cells:

$$\Omega_{\boldsymbol{\Lambda},\boldsymbol{z}}=\bigcap_{s=1}^n\Omega_{\lambda^{(s)}}(\mathcal{F}(z_s)).$$

Note that due to our assumption, $\Omega_{\boldsymbol{\Lambda},\boldsymbol{z}}$ is a finite subset of $\mathrm{Gr}(N,d)$. Note also that $\Omega_{\boldsymbol{\Lambda},\boldsymbol{z}}$ is non-empty if and only if $(\otimes_{i=1}^nV_{\lambda^{(i)}})^{\mathfrak{sl}_N}\neq 0$.

The $\mathfrak{gl}_N$-stratification of $\mathrm{Gr}(N,d)$

Define a partial order $\geqslant$ on the set of sequences of partitions with at most N parts as follows. Let $\boldsymbol{\Lambda}=(\lambda^{(1)},\dots,\lambda^{(n)})$, $\boldsymbol{\Xi}=(\xi^{(1)},\dots,\xi^{(m)})$ be two sequences of partitions with at most N parts. We say that $\boldsymbol{\Lambda}\geqslant\boldsymbol{\Xi}$ if there exists a partition $\{I_1,\dots,I_m\}$ of the set $\{1,2,\dots,n\}$ such that

$$\mathrm{Hom}_{\mathfrak{gl}_N}\Big(V_{\xi^{(i)}},\bigotimes_{j\in I_i}V_{\lambda^{(j)}}\Big)\neq 0,\qquad i=1,\dots,m.$$

We say that $\boldsymbol{\Lambda}=(\lambda^{(1)},\dots,\lambda^{(n)})$ is *d-nontrivial* if $(\otimes_{i=1}^nV_{\lambda^{(i)}})^{\mathfrak{sl}_N}\neq 0$ and $|\lambda^{(s)}|>0$, $s=1,\dots,n$, and $|\boldsymbol{\Lambda}|=N(d-N)$.

Define $\Omega_{\boldsymbol{\Lambda}}$ by the formula

$$\Omega_{\boldsymbol{\Lambda}}:=\bigcup_{\boldsymbol{z}\in\mathring{\mathbb{P}}_n}\Omega_{\boldsymbol{\Lambda},\boldsymbol{z}}\subset\mathrm{Gr}(N,d).$$

Theorem 1

We have

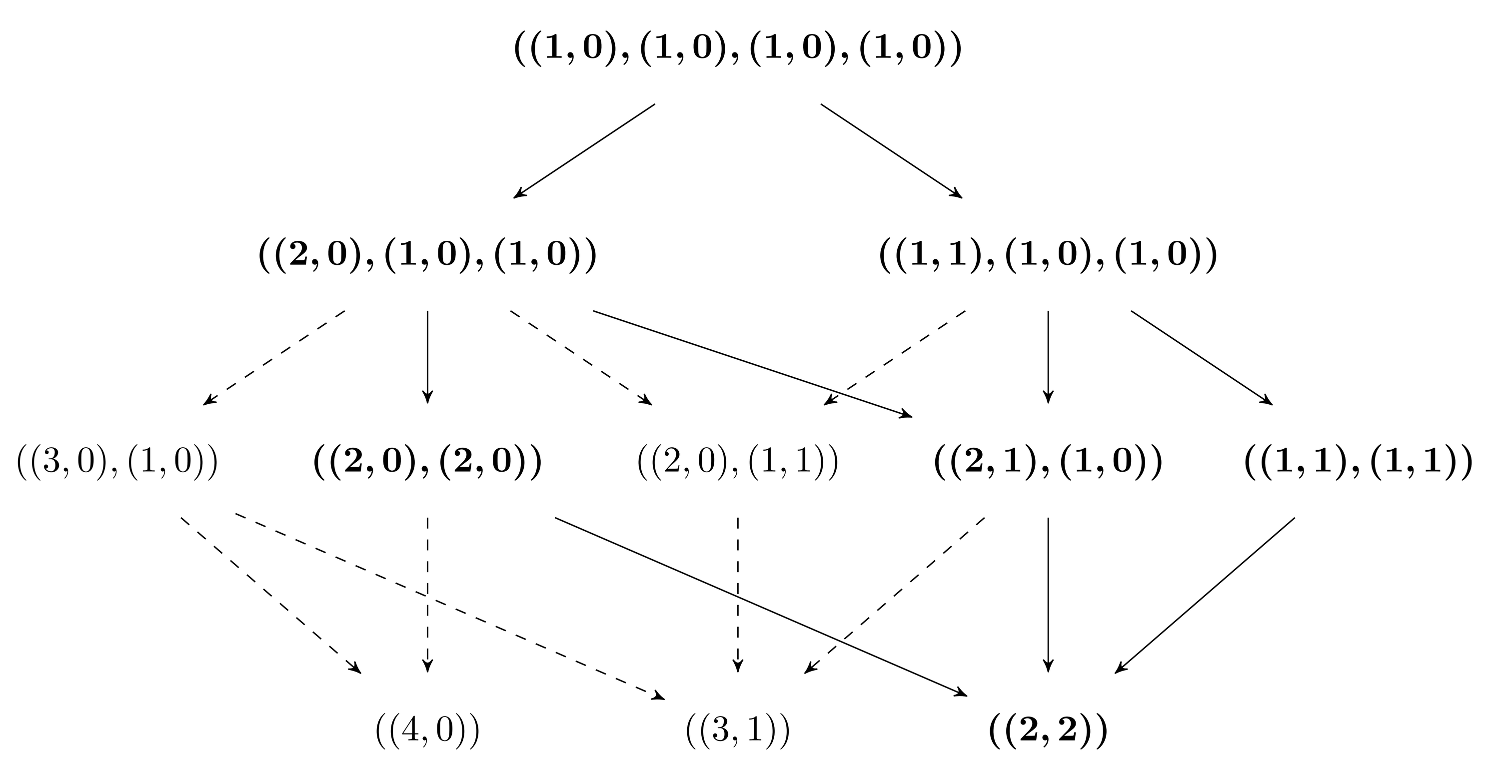
$$\mathrm{Gr}(N,d)=\bigsqcup_{d\text{-nontrivial }\boldsymbol{\Lambda}}\Omega_{\boldsymbol{\Lambda}}.$$

For d-nontrivial $\boldsymbol{\Lambda}$, we have

$$\overline{\Omega}_{\boldsymbol{\Lambda}}=\bigsqcup_{\substack{\boldsymbol{\Xi}\leqslant\boldsymbol{\Lambda},\\ d\text{-nontrivial }\boldsymbol{\Xi}}}\Omega_{\boldsymbol{\Xi}}.$$

The theorem implies that the subsets $\Omega_{\boldsymbol{\Lambda}}$ with d-nontrivial $\boldsymbol{\Lambda}$ give a stratification of $\mathrm{Gr}(N,d)$. We call it the *$\mathfrak{gl}_N$-stratification of $\mathrm{Gr}(N,d)$*.

Example. We give an example of the $\mathfrak{gl}_2$-stratification for $\mathrm{Gr}(2,4)$ in the following picture.



Self-dual Grassmannian $\mathrm{sGr}(N,d)$

Let $X\in\mathrm{Gr}(N,d)$. Define $X^\vee$ be the N-dimensional space of polynomials by the formula:

$$X^\vee=\{\det\big(d^if_j/dx^i\big)_{i,j=1}^{N-1},\ f_j(x)\in X\}.$$

We call X *self-dual* if $X^\vee=g\cdot X$ for some polynomial $g(x)$. Define the *self-dual Grassmannian* $\mathrm{sGr}(N,d)$ as the subset of $\mathrm{Gr}(N,d)$ of all self-dual spaces.

Denote by $\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{z}}$ the set of all self-dual spaces in $\Omega_{\boldsymbol{\Lambda},\boldsymbol{z}}$ and by $\mathrm{s}\Omega_{\boldsymbol{\Lambda}}$ the set of all self-dual spaces in $\Omega_{\boldsymbol{\Lambda}}$:

$$\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{z}}=\Omega_{\boldsymbol{\Lambda},\boldsymbol{z}}\bigcap\mathrm{sGr}(N,d)\quad\text{and}\quad\mathrm{s}\Omega_{\boldsymbol{\Lambda}}=\Omega_{\boldsymbol{\Lambda}}\bigcap\mathrm{sGr}(N,d).$$

Set $\mathfrak{g}_{2r+1}=\mathfrak{sp}_{2r}$ and $\mathfrak{g}_{2r}=\mathfrak{so}_{2r+1}$. Let μ be a dominant integral $\mathfrak{g}_N$-weight and $k\in\mathbb{Z}_{\geqslant 0}$. Define a partition $\mu_{A,k}$ with at most N parts by the rule: $(\mu_{A,k})_N=k$ and

$$(\mu_{A,k})_i-(\mu_{A,k})_{i+1}=\begin{cases}\langle\mu,\check\alpha_i\rangle, & \text{if } 1\leqslant i\leqslant[\frac N2],\\ \langle\mu,\check\alpha_{N-i}\rangle, & \text{if } [\frac N2]<i\leqslant N-1.\end{cases}$$

We call $\mu_{A,k}$ the partition *associated with weight μ and integer k*.

Let $\boldsymbol{\Lambda}=(\lambda^{(1)},\dots,\lambda^{(n)})$ be a sequence of dominant integral $\mathfrak{g}_N$-weights and let $\boldsymbol{k}=(k_1,\dots,k_n)$ be an n-tuple of nonnegative integers. Then denote $\boldsymbol{\Lambda}_{A,\boldsymbol{k}}=(\lambda^{(1)}_{A,k_1},\dots,\lambda^{(n)}_{A,k_n})$ the sequence of partitions associated with $\lambda^{(s)}$ and k_s, $s=1,\dots,n$. We write $\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k}}$ for $\mathrm{s}\Omega_{\boldsymbol{\Lambda}_{A,\boldsymbol{k}}}$ and $\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k},\boldsymbol{z}}$ for $\mathrm{s}\Omega_{\boldsymbol{\Lambda}_{A,\boldsymbol{k}},\boldsymbol{z}}$.

The $\mathfrak{g}_N$-stratification of $\mathrm{sGr}(N,d)$

Define a partial order $\geqslant$ on the set of pairs $\{(\boldsymbol{\Lambda},\boldsymbol{k})\}$ as follows. Let $\boldsymbol{\Lambda}=(\lambda^{(1)},\dots,\lambda^{(n)})$, $\boldsymbol{\Xi}=(\xi^{(1)},\dots,\xi^{(m)})$ be two sequences of dominant integral $\mathfrak{g}_N$-weights. Let $\boldsymbol{k}=(k_1,\dots,k_n)$, $\boldsymbol{l}=(l_1,\dots,l_m)$ be two tuples of nonnegative integers. We say that $(\boldsymbol{\Lambda},\boldsymbol{k})\geqslant(\boldsymbol{\Xi},\boldsymbol{l})$ if there exists a partition $\{I_1,\dots,I_m\}$ of $\{1,2,\dots,n\}$ such that

$$\mathrm{Hom}_{\mathfrak{g}_N}\Big(V_{\xi^{(i)}},\bigotimes_{j\in I_i}V_{\lambda^{(j)}}\Big)\neq 0,\qquad |\xi^{(i)}_{A,l_i}|=\sum_{j\in I_i}|\lambda^{(j)}_{A,k_j}|,\qquad i=1,\dots,m.$$

We say that $(\boldsymbol{\Lambda},\boldsymbol{k})$ is *d-nontrivial* if and only if $(\otimes_{i=1}^nV_{\lambda^{(i)}})^{\mathfrak{g}_N}\neq 0$, $|\lambda^{(s)}_{A,k_s}|>0$, $s=1,\dots,n$, and $|\boldsymbol{\Lambda}_{A,\boldsymbol{k}}|=N(d-N)$.

Theorem 2

We have

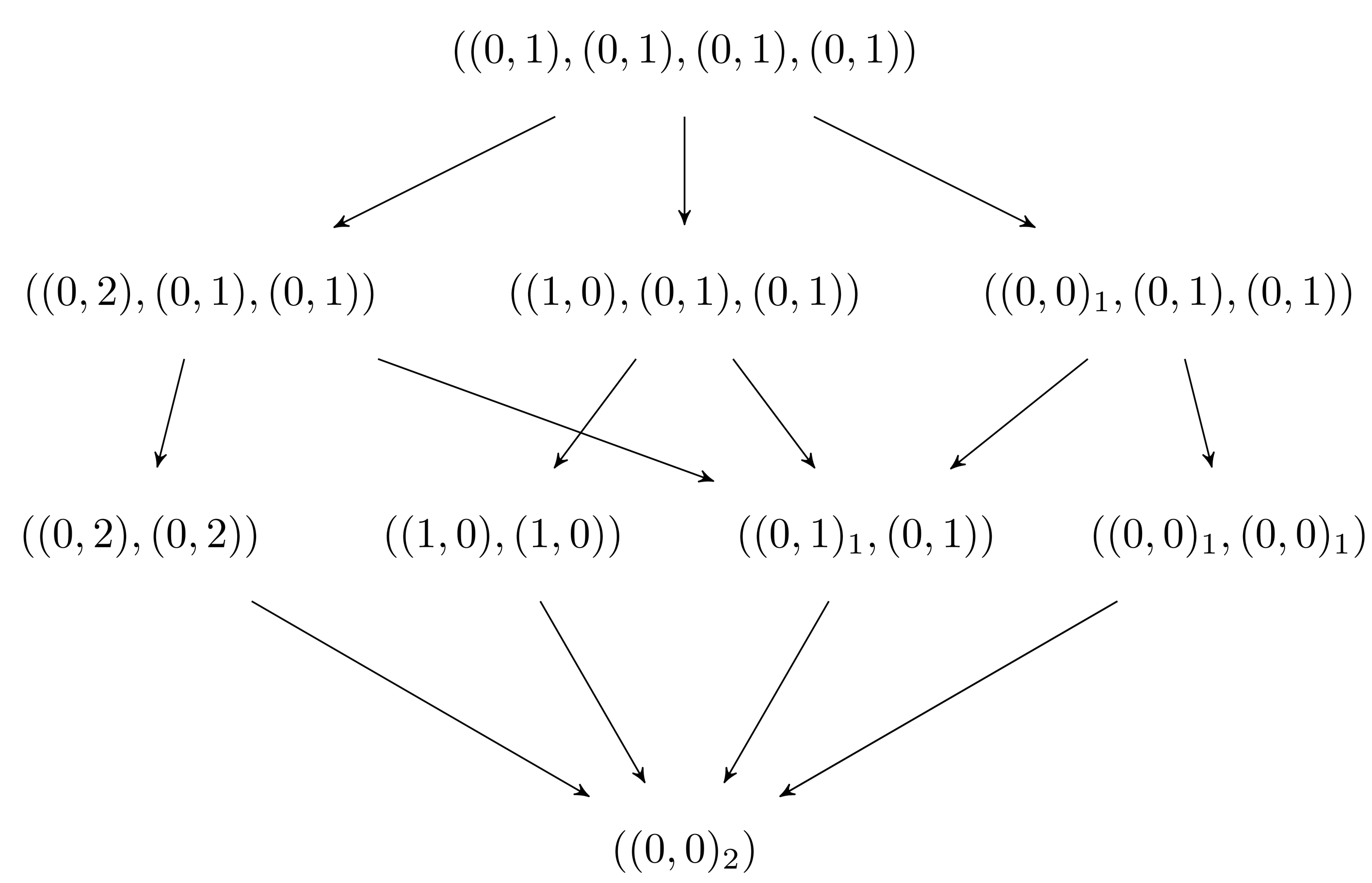
$$\mathrm{sGr}(N,d)=\bigsqcup_{d\text{-nontrivial }(\boldsymbol{\Lambda},\boldsymbol{k})}\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k}}.$$

For d-nontrivial $(\boldsymbol{\Lambda},\boldsymbol{k})$, we have

$$\overline{\mathrm{s}\Omega}_{\boldsymbol{\Lambda},\boldsymbol{k}}=\bigsqcup_{\substack{(\boldsymbol{\Xi},\boldsymbol{l})\leqslant(\boldsymbol{\Lambda},\boldsymbol{k}),\\ d\text{-nontrivial }(\boldsymbol{\Xi},\boldsymbol{l})}}\mathrm{s}\Omega_{\boldsymbol{\Xi},\boldsymbol{l}}.$$

The theorem implies that the subsets $\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k}}$ with d-nontrivial $(\boldsymbol{\Lambda},\boldsymbol{k})$ give a stratification of $\mathrm{sGr}(N,d)$, similar to the $\mathfrak{gl}_N$-stratification of $\mathrm{Gr}(N,d)$. We call it the *$\mathfrak{g}_N$-stratification of $\mathrm{sGr}(N,d)$*.

Example. The following picture gives an example for $\mathfrak{so}_5$-stratification of $\mathrm{sGr}(4,6)$.



Acknowledgements

The authors thank V. Chari, A. Gabrielov, and L. Rybnikov for useful discussions.