JACOBI-TRUDI IDENTITY AND DRINFELD FUNCTOR FOR SUPER YANGIAN

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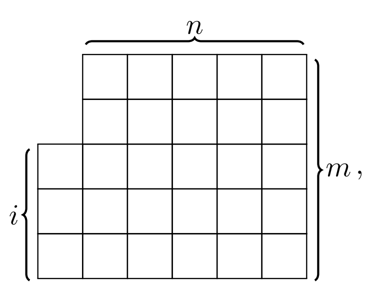
ABSTRACT. We show that the quantum Berezinian which gives a generating function of the integrals of motions of XXX spin chains associated to super Yangian $\mathrm{Y}(\mathfrak{gl}_{m|n})$ can be written as a ratio of two difference operators of orders m and n whose coefficients are ratios of transfer matrices corresponding to explicit skew Young diagrams.

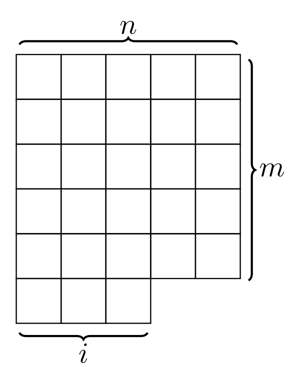
In the process, we develop several missing parts of the representation theory of $\mathrm{Y}(\mathfrak{gl}_{m|n})$ such as q-character theory, Jacobi-Trudi identity, Drinfeld functor, extended T-systems, Harish-Chandra map.

Keywords: super Yangian, Jacobi-Trudi identity, degenerate affine Hecke algebra, Drinfeld functor, T-systems, transfer matrices.

1. INTRODUCTION

This paper deals with the representation theory of $\mathrm{Y}(\mathfrak{gl}_{m|n})$, the Yangian associated with the general Lie superalgebra $\mathfrak{gl}_{m|n}$. However, the original motivation comes from the theory of integrable systems. The superalgebra $\mathrm{Y}(\mathfrak{gl}_{m|n})$ contains a commutative subalgebra (called the *Bethe subalgebra*) given by the expansion of a certain quantum Berezinian, see [MR14] and (5.1). The coefficients of the expansion are transfer matrices related to representations associated to single-column Young diagrams, see (5.2). On the other hand, the method of Bethe ansatz suggests that it is natural to expect that the quantum Berezinian can be compactly written as in the form $\mathfrak{D}_1\mathfrak{D}_2^{-1}$ where $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are difference operators of orders m and n respectively, see [HMVY19, HLM19]. We show that this is indeed the case and, moreover, the i-th coefficients of $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are given by transfer matrices corresponding to skew Young diagrams





respectively, divided by the transfer matrix corresponding to the $m \times n$ rectangle, see Theorem 5.12. That explains the formulas for the rational difference operators (5.9) corresponding to solutions of Bethe ansatz equations (5.8) and makes a step towards understanding the so-called $\mathfrak{gl}_{m|n}$ spaces of rational functions [HMVY19], see the discussion in Section 5.5.

The proof of the above result is obtained by a direct computation with the use of the Jacobi-Trudi identity for the $\mathrm{Y}(\mathfrak{gl}_{m|n})$-modules related to the participating skew Young diagrams.

More generally, we study irreducible $\mathrm{Y}(\mathfrak{gl}_{m|n})$-modules $L(\lambda/\mu)$ corresponding to arbitrary skew Young diagrams. These are supersymmetric analogs of the tame modules [NT98], also known as snake modules

[MY12a]. We establish the formulas for the q-characters (that is the joint generalized eigenvalues of the Gelfand-Tsetlin subalgebra) of such modules in terms of the semi-standard Young tableaux, see Theorem 3.4, and prove the Jacobi-Trudi identity for the q-characters. The ways to attack the Jacobi-Trudi identity are well-known. Here we use the Lindström-Gessel-Viennot method with the appropriate modifications, see Theorem 3.16.

We take the opportunity to define the Drinfeld functor which constructs a $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module from a representation of the degenerate affine Hecke algebra $\mathcal{H}_l$. The key fact is that the Jacobi-Trudi formula does not depend on m and n. It implies that the $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module $L(\lambda/\mu)$ comes from the same $\mathcal{H}_l$-module for all m and n. Since the Drinfeld functor is exact and an equivalence of categories in the even case, see [Dri86, CP96] or Theorem 4.3, we obtain a tool to translate the information about representations of even Yangian $\mathrm{Y}(\mathfrak{gl}_N)$ to super Yangian $\mathrm{Y}(\mathfrak{gl}_{m|n})$.

We give a couple of such examples, describing sufficient conditions for tensor products of evaluation $\mathrm{Y}(\mathfrak{gl}_{m|n})$-modules to be irreducible, see Theorem 4.18 and establishing the extended T-systems, see Corollary 4.24.

The paper is constructed as follows. We start by organizing known facts about the general Lie superalgebra $\mathfrak{gl}_{m|n}$ and the super Yangian $\mathrm{Y}(\mathfrak{gl}_{m|n})$ in Section 2. In addition we compute some information about the coproduct, see Proposition 2.7, which allows us to introduce the q-character ring homomorphism in Section 2.7. Section 3 is devoted to the study of $\mathrm{Y}(\mathfrak{gl}_{m|n})$-modules related to skew Young diagrams. Drinfeld functor is defined and studied in Section 4. The applications in the form of irreducibility conditions for tensor products and the extended T-systems are given in Sections 4.5 and 4.6, respectively. Section 5 deals with transfer matrices and, in particular, with quantum Berezinians. We study the Harish-Chandra map which connects the transfer matrices to q-characters, see Section 5.3. The main result of this section is Theorem 5.12.

Acknowledgments. We thank V. Tarasov for stimulating discussions. This work was partially supported by a grant from the Simons Foundation #353831.

2. Super Yangian $\mathrm{Y}(\mathfrak{gl}_{m|n})$

2.1. **Lie superalgebra $\mathfrak{gl}_{m|n}$.** Through out the paper, we work over $\mathbb{C}$. In this section, we recall the basics of the Lie superalgebra $\mathfrak{gl}_{m|n}$, see e.g. [CW12] for more detail. We simply write $\mathfrak{gl}_m$ for $\mathfrak{gl}_{m|0}$.

A *vector superspace* $W = W_{\bar 0} \oplus W_{\bar 1}$ is a $\mathbb{Z}_2$-graded vector space. We call elements of $W_{\bar 0}$ *even* and elements of $W_{\bar 1}$ *odd*. We write $|w| \in \{\bar 0, \bar 1\}$ for the parity of a homogeneous element $w \in W$. Set $(-1)^{\bar 0} = 1$ and $(-1)^{\bar 1} = -1$.

Fix $m, n \in \mathbb{Z}_{\geqslant 0}$. Set $I := \{1, 2, \dots, m+n-1\}$ and $\bar I := \{1, 2, \dots, m+n\}$. We also set $|i| = \bar 0$ for $1 \leqslant i \leqslant m$ and $|i| = \bar 1$ for $m < i \leqslant m+n$. Define $s_i = (-1)^{|i|}$ for $i \in \bar I$.

The Lie superalgebra $\mathfrak{gl}_{m|n}$ is generated by elements e_{ij}, $i, j \in \bar I$, with the supercommutator relations

$$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - (-1)^{(|i|+|j|)(|k|+|l|)} \delta_{il} e_{kj},$$

where the parity of e_{ij} is $|i| + |j|$. Denote by $\mathrm{U}(\mathfrak{gl}_{m|n})$ the universal enveloping superalgebra of $\mathfrak{gl}_{m|n}$. The superalgebra $\mathrm{U}(\mathfrak{gl}_{m|n})$ is a Hopf superalgebra with the coproduct given by $\Delta(x) = 1 \otimes x + x \otimes 1$ for all $x \in \mathfrak{gl}_{m|n}$.

The *Cartan subalgebra* $\mathfrak{h}$ of $\mathfrak{gl}_{m|n}$ is spanned by e_{ii}, $i \in \bar I$. Let ϵ_i, $i \in \bar I$, be a basis of $\mathfrak{h}^*$ (the dual space of $\mathfrak{h}$) such that $\epsilon_i(e_{jj}) = \delta_{ij}$. There is a bilinear form $(\, , \,)$ on $\mathfrak{h}^*$ given by $(\epsilon_i, \epsilon_j) = s_i \delta_{ij}$. The *root system* $\boldsymbol{\Phi}$ is a subset of $\mathfrak{h}^*$ given by

$$\boldsymbol{\Phi} := \{\epsilon_i - \epsilon_j \mid i, j \in \bar I \text{ and } i \neq j\}.$$

We call a root $\epsilon_i - \epsilon_j$ *even* (resp. *odd*) if $|i| = |j|$ (resp. $|i| \neq |j|$).

Set $\alpha_i := \epsilon_i - \epsilon_{i+1}$ for $i \in I$. Denote by $\mathbf{P} := \oplus_{i\in\bar{I}}\mathbb{Z}\epsilon_i$, $\mathbf{Q} := \oplus_{i\in I}\mathbb{Z}\alpha_i$, and $\mathbf{Q}_{\geqslant 0} := \oplus_{i\in I}\mathbb{Z}_{\geqslant 0}\alpha_i$ the *weight lattice*, the *root lattice*, and the *cone of positive roots*, respectively. Define a partial ordering $\geqslant$ on $\mathfrak{h}^*$: $\alpha \geqslant \beta$ if $\alpha - \beta \in \mathbf{Q}_{\geqslant 0}$. There is a natural $\mathbb{Z}_2$-grading on $\mathbf{P}$ such that the parity p_α of $\alpha \in \mathbf{P}$ is given by

$$p_\alpha := \sum_{i\in\bar{I}} (\alpha, \epsilon_i)|i|. \tag{2.1}$$

A module M over a superalgebra $\mathcal{A}$ is a vector superspace M with the homomorphism of superalgebras $\mathcal{A} \to \mathrm{End}(M)$. A $\mathfrak{gl}_{m|n}$-module is a module over $\mathrm{U}(\mathfrak{gl}_{m|n})$.

Let $\alpha \in \mathbf{P}$. We call a nonzero vector v in a $\mathfrak{gl}_{m|n}$-module M a *singular vector of weight* α if v satisfies

$$e_{ii}v = \alpha(e_{ii})v, \quad e_{jk}v = 0,$$

for $i \in \bar{I}$ and $1 \leqslant j < k \leqslant m+n$. Denote by $L^p(\alpha)$ the irreducible $\mathfrak{gl}_{m|n}$-module generated by a singular vector of weight α and of parity p. We simply write $L(\alpha)$ for $L^{p_\alpha}(\alpha)$.

For a $\mathfrak{gl}_{m|n}$-module M, define the *weight subspace of weight* α by

$$(M)_\alpha := \{v \in M \mid e_{ii}v = \alpha(e_{ii})v,\ i \in \bar{I}\}.$$

If $(M)_\alpha \neq 0$, we call α a *weight* of M. Denote by $\mathrm{wt}(M)$ the set of all weights of M. If $v \in (M)_\alpha$ and v is non-zero, then we write $\mathrm{wt}(v) = \alpha$. We focus on the $\mathfrak{gl}_{m|n}$-modules M such that $(M)_\alpha = 0$ unless $\alpha \in \mathbf{P}$. We say that M is $\mathbf{P}$*-graded*. We call a vector $v \in M$ *singular* if $e_{ij}v = 0$ for $1 \leqslant i < j \leqslant m+n$.

Let $V := \mathbb{C}^{m|n}$ be the vector superspace with a basis v_i, $i \in \bar{I}$, such that $|v_i| = |i|$. Let $E_{ij} \in \mathrm{End}(V)$ be the linear operators such that $E_{ij}v_k = \delta_{jk}v_i$. The map $\rho_V : \mathfrak{gl}_{m|n} \to \mathrm{End}(V)$, $e_{ij} \mapsto E_{ij}$ defines a $\mathfrak{gl}_{m|n}$-module structure on V. As a $\mathfrak{gl}_{m|n}$-module, V is isomorphic to $L(\epsilon_1)$. The vector v_i has weight ϵ_i. The highest weight vector is v_1 and the lowest weight vector is v_{m+n}.We call it the *vector representation* of $\mathfrak{gl}_{m|n}$.

Let $\mathfrak{gl}_{n|m}$ be the Lie superalgebra defined in the same way as $\mathfrak{gl}_{m|n}$ with m and n interchanged. There exists a Lie superalgebra isomorphism between $\mathfrak{gl}_{m|n}$ and $\mathfrak{gl}_{n|m}$ given by the map

$$\varsigma_{m|n} : e_{ij} \mapsto e_{m+n+1-i,m+n+1-j}. \tag{2.2}$$

Fix $m', n' \in \mathbb{Z}_{\geqslant 0}$ and consider the Lie superalgebra $\mathfrak{gl}_{m'|n'}$. For this algebra we also choose the standard parity, namely, we set $|i| = \bar{0}$ if and only if $1 \leqslant i \leqslant m'$.

We also consider a larger Lie superalgebra $\mathfrak{gl}_{m'+m|n'+n}$. For $\mathfrak{gl}_{m'+m|n'+n}$, we fix the parity by

$$|i| = \begin{cases} \bar{0}, & \text{if } 1 \leqslant i \leqslant m' \text{ or } m'+n'+1 \leqslant i \leqslant m'+n'+m, \\ \bar{1}, & \text{if } m'+1 \leqslant i \leqslant m'+n' \text{ or } m'+n'+m+1 \leqslant i \leqslant m'+n'+m+n. \end{cases} \tag{2.3}$$

Clearly, we have the embeddings of Lie superalgebras given by

$$\mathfrak{gl}_{m'|n'} \to \mathfrak{gl}_{m'+m|n'+n}, \quad e_{ij} \mapsto e_{ij},$$

$$\mathfrak{gl}_{m|n} \to \mathfrak{gl}_{m'+m|n'+n}, \quad e_{ij} \mapsto e_{m'+n'+i,m'+n'+j}.$$

Define the *supertrace* $\mathrm{str} : \mathrm{End}(\mathbb{C}^{m|n}) \to \mathbb{C}$,

$$\mathrm{str}(E_{ij}) = s_j\delta_{ij}.$$

The supertrace is supercyclic, that is

$$\mathrm{str}([E_{ij}, E_{rs}]) = 0.$$

Here $[\cdot,\cdot]$ is the supercommutator of linear operators.

Denote by $\mathfrak{sl}_{m|n}$ the Lie subalgebra of $\mathfrak{gl}_{m|n}$ consisting of all elements acting on V as matrices with zero supertrace.

Define the *supertranspositions* t and $\top$,

$$t : \mathrm{End}(V) \to \mathrm{End}(V), \quad E_{ij}^{t} = (-1)^{|i||j|+|j|} E_{ji}, \tag{2.4}$$

$$\top : \mathrm{End}(V) \to \mathrm{End}(V), \quad E_{ij}^{\top} = (-1)^{|i||j|+|i|} E_{ji}. \tag{2.5}$$

Both supertranspositions are anti-homomorphisms and respect the supertrace,

$$(AB)^* = (-1)^{|A||B|} B^* A^*, \qquad \mathrm{str}(A) = \mathrm{str}(A^*), \tag{2.6}$$

for all $(m+n)\times(m+n)$ matrices A and B, where $*$ is either t or $\top$. We also have $t = \top^3$ and $t^4 = \top^4 = 1$.

2.2. **Hook partitions, skew Young diagrams, and polynomial modules.** Let $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots)$ be a partition of ℓ: $\lambda_i \in \mathbb{Z}_{\geqslant 0}$, $\lambda_i = 0$ if $i \gg 0$, and $|\lambda| := \sum_{i=1}^{\infty} \lambda_i = \ell$. We denote by λ' the *conjugate* of the partition λ. The number λ_1' is the length of the partition λ, namely the number of nonzero parts of λ. Let $\mu = (\mu_1 \geqslant \mu_2 \geqslant \cdots)$ be another partition such that $\mu_i \leqslant \lambda_i$ for all $i = 1, 2, \ldots$. Consider the *skew Young diagram* λ/μ which is defined as the set of pairs

$$\{(i,j) \in \mathbb{Z}^2 \mid i \geqslant 1,\ \lambda_i \geqslant j > \mu_i\}.$$

When μ is the zero partition, then λ/μ is the usual Young diagram corresponding to λ.

We use the standard representation of skew Young diagrams on the coordinate plane $\mathbb{R}^2$ with coordinates (x, y). Here we use the convention that x increases from north to south while y increases from west to east. Moreover, the pair $(i,j) \in \lambda/\mu$ is represented by the unit box whose south-eastern corner has coordinate $(i,j) \in \mathbb{Z}^2$. We also define the *content* of the box corresponding to $(i,j) \in \lambda/\mu$ by $c(i,j) = j - i$.

A *semi-standard Young tableau* of shape λ/μ is the skew Young diagram λ/μ with an element from $\{1, 2, \ldots, m+n\}$ inserted in each box such that the following conditions are satisfied:

(i) the numbers in boxes are weakly increasing along rows and columns;
(ii) the numbers from $\{1, 2, \ldots, m\}$ are strictly increasing along columns;
(iii) the numbers from $\{m+1, m+2, \ldots, m+n\}$ are strictly increasing along rows.

For a semi-standard Young tableau $\mathcal{T}$ of shape λ/μ, denote by $\mathcal{T}(i,j)$ the number in the box representing the pair $(i,j) \in \lambda/\mu$.

Example 2.1. Let $\lambda = (5,3,3,3,3)$, $\mu = (3,3,2,2)$, $m = n = 2$, then the skew Young diagram λ/μ has the shape as one of the following.

			1	2
		3		
		3		
2	3	4		

			3	4
		0		
		-1		
-4	-3	-2		

In the picture above, the left one is an example of a semi-standard Young tableau of shape λ/μ. We have $\mathcal{T}(1,4) = 1$, $\mathcal{T}(1,5) = \mathcal{T}(5,1) = 2$, $\mathcal{T}(3,3) = \mathcal{T}(4,3) = \mathcal{T}(5,2) = 3$, and $\mathcal{T}(5,3) = 4$. In the right, we wrote in each box its content. $\square$

A *standard Young tableau* of shape λ/μ is the skew Young diagram λ/μ with an element from $\{1, \ldots, |\lambda| - |\mu|\}$ inserted in each box such that the numbers in boxes are strictly increasing along rows and columns.

Example 2.2. There is a distinguished standard Young tableau obtained by filling numbers along rows from left to right and top to bottom. We call it the *row tableau*. Similarly, one defines the *column tableau*. Here are the row (on the left) and the column (on the right) tableaux for the skew Young diagram λ/μ in the previous example.

$\square$

Recall that $V = \mathbb{C}^{m|n}$ denotes the vector representation of $\mathfrak{gl}_{m|n}$. A $\mathfrak{gl}_{m|n}$-module is called a *polynomial module* if it is a submodule of $V^{\otimes l}$ for some $l \in \mathbb{Z}_{\geqslant 0}$. We call a partition λ an $(m|n)$-*hook partition* if $\lambda_{m+1} \leqslant n$. Let $\mathcal{P}_l(m|n)$ be the set of all $(m|n)$-hook partitions of l and $\mathcal{P}(m|n)$ the set of all $(m|n)$-hook partitions. In particular, $\mathcal{P}_l(m) := \mathcal{P}_l(m|0)$ is the set of all partitions of l with length $\leqslant m$. It is well-known that irreducible polynomial $\mathfrak{gl}_{m|n}$-modules are parameterized by $\mathcal{P}(m|n)$.

For $\lambda \in \mathcal{P}(m|n)$, define the $\mathfrak{gl}_{m|n}$-weight $\lambda^\natural$ by

$$\lambda^\natural = \sum_{i=1}^{m} \lambda_i \epsilon_i + \sum_{j=1}^{n} \max\{\lambda'_j - m, 0\} \epsilon_{m+j}. \tag{2.7}$$

We sometimes use the notation $\lambda^\sharp_{[m|n]}$ to stress the dependence of $\lambda^\sharp$ on m and n.

Let $\mathcal{P} \in \mathrm{End}(V \otimes V)$ be the super flip operator,

$$\mathcal{P} = \sum_{i,j \in \bar{I}} s_j E_{ij} \otimes E_{ji}.$$

Let $\mathfrak{S}_l$ be the symmetric group permuting $\{1, 2, \ldots, l\}$. The symmetric group $\mathfrak{S}_l$ acts naturally on $V^{\otimes l}$, where the simple transposition $\sigma_k = (k, k+1)$ acts as

$$\mathcal{P}^{(k,k+1)} = \sum_{i,j \in \bar{I}} s_j E_{ij}^{(k)} E_{ji}^{(k+1)} \in \mathrm{End}(V^{\otimes l}), \tag{2.8}$$

where we use the standard notation

$$E_{ij}^{(k)} = 1^{\otimes (k-1)} \otimes E_{ij} \otimes 1^{\otimes (l-k)} \in \mathrm{End}(V^{\otimes l}), \qquad k = 1, \ldots, l.$$

Let $\mathcal{S}(\lambda)$ be the finite-dimensional irreducible representation of $\mathfrak{S}_l$ corresponding to the partition λ.

Theorem 2.3 (Schur-Sergeev duality [Ser85])**.** *The $\mathfrak{S}_l$-action and $\mathfrak{gl}_{m|n}$-action on $V^{\otimes l}$ commute. Moreover, as a $\mathrm{U}(\mathfrak{gl}_{m|n}) \otimes \mathbb{C}[\mathfrak{S}_l]$-module, we have*

$$V^{\otimes l} \cong \bigoplus_{\lambda \in \mathcal{P}_l(m|n)} L(\lambda^\natural) \otimes \mathcal{S}(\lambda). \qquad \square$$

For $\lambda \in \mathcal{P}(m|n)$, we have $\lambda' \in \mathcal{P}(n|m)$. The $\mathfrak{gl}_{m|n}$-module obtained by pulling back the $\mathfrak{gl}_{n|m}$-module $L((\lambda')^\sharp_{[n|m]})$ through the isomorphism $\varsigma : \mathfrak{gl}_{m|n} \to \mathfrak{gl}_{n|m}$, see (2.2), is isomorphic to $L(\lambda^\sharp_{[m|n]})$.

For $\lambda \in \mathcal{P}(m'+m|n'+n)$, define a $\mathfrak{gl}_{m'+m|n'+n}$-weight λ° by

$$\lambda^\circ = \sum_{i=1}^{m'} \lambda_i \epsilon_i + \sum_{j=1}^{n'} \max\{\lambda'_j - m', 0\}\epsilon_{m'+j} + \sum_{i=m'+1}^{m'+m} \max\{\lambda_i - n', 0\}\epsilon_{n'+i} + \sum_{j=n'+1}^{n'+n} \max\{\lambda'_j - m' - m, 0\}\epsilon_{m'+m+j}. \tag{2.9}$$

This definition is dictated by parity (2.3) we chose, see [BR83]. We will use λ° in Section 3 to define skew representations of super Yangian.

2.3. **Super Yangian** $\mathrm{Y}(\mathfrak{gl}_{m|n})$**.** We recall the definition of super Yangian $\mathrm{Y}(\mathfrak{gl}_{m|n})$ from [Naz91].

The super Yangian $\mathrm{Y}(\mathfrak{gl}_{m|n})$ is the $\mathbb{Z}_2$-graded unital associative algebra over $\mathbb{C}$ with generators $\{t_{ij}^{(r)} \mid i,j \in \bar{I},\, r \geqslant 1\}$ and defining relations

$$[t_{ij}^{(r)}, t_{kl}^{(s)}] = (-1)^{|i||j|+|i||k|+|j||k|} \sum_{a=0}^{\min(r,s)-1} (t_{kj}^{(a)} t_{il}^{(r+s-1-a)} - t_{kj}^{(r+s-1-a)} t_{il}^{(a)}), \tag{2.10}$$

where the generators $t_{ij}^{(r)}$ have parities $|i|+|j|$.

The super Yangian $\mathrm{Y}(\mathfrak{gl}_{m|n})$ has the RTT presentation as follows. Define the rational R-matrix $R(u) \in \mathrm{End}(V \otimes V)$ by $R(u) = 1 - \mathcal{P}/u$. The rational R-matrix satisfies the quantum Yang-Baxter equation

$$R_{12}(u_1-u_2)R_{13}(u_1-u_3)R_{23}(u_2-u_3) = R_{23}(u_2-u_3)R_{13}(u_1-u_3)R_{12}(u_1-u_2). \tag{2.11}$$

Define the generating series

$$t_{ij}(u) = \delta_{ij} + \sum_{k=1}^{\infty} t_{ij}^{(k)} u^{-k}$$

and the operator $T(u) \in \mathrm{End}(V) \otimes \mathrm{Y}(\mathfrak{gl}_{m|n})[[u^{-1}]]$,

$$T(u) = \sum_{i,j \in \bar{I}} (-1)^{|i||j|+|j|} E_{ij} \otimes t_{ij}(u).$$

Denote by

$$T_k(u) = \sum_{i,j \in \bar{I}} (-1)^{|i||j|+|j|} E_{ij}^{(k)} \otimes t_{ij}(u) \in \mathrm{End}(V^{\otimes l}) \otimes \mathrm{Y}(\mathfrak{gl}_{m|n})[[u^{-1}]]. \tag{2.12}$$

Then defining relations (2.10) can be written as

$$R(u_1-u_2)T_1(u_1)T_2(u_2) = T_2(u_2)T_1(u_1)R(u_1-u_2) \in \mathrm{End}(V^{\otimes 2}) \otimes \mathrm{Y}(\mathfrak{gl}_{m|n})[[u^{-1}]],$$

In terms of generating series, defining relations (2.10) are equivalent to

$$(u_1-u_2)[t_{ij}(u_1), t_{kl}(u_2)] = (-1)^{|i||j|+|i||k|+|j||k|}(t_{kj}(u_1)t_{il}(u_2) - t_{kj}(u_2)t_{il}(u_1)). \tag{2.13}$$

The super Yangian $\mathrm{Y}(\mathfrak{gl}_{m|n})$ is a Hopf superalgebra with coproduct, antipode, counit given by

$$\Delta : t_{ij}(u) \mapsto \sum_{k \in \bar{I}} t_{ik}(u) \otimes t_{kj}(u), \qquad S : T(u) \mapsto T(u)^{-1}, \qquad \varepsilon : T(u) \mapsto 1. \tag{2.14}$$

Let Δ^{op} be the opposite coproduct of $\mathrm{Y}(\mathfrak{gl}_{m|n})$,

$$\Delta^{\mathrm{op}}(t_{ij}(u)) = \sum_{k \in \bar{I}} (-1)^{(|i|+|k|)(|j|+|k|)} t_{kj}(u) \otimes t_{ik}(u). \tag{2.15}$$

For $z \in \mathbb{C}$ there exists an isomorphism of Hopf superalgebras,

$$\tau_z : \mathrm{Y}(\mathfrak{gl}_{m|n}) \to \mathrm{Y}(\mathfrak{gl}_{m|n}), \qquad t_{ij}(u) \mapsto t_{ij}(u-z). \tag{2.16}$$

For any $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module M, denote by M_z the $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module obtained from pulling back M through the isomorphism τ_z.

The super Yangian $\mathrm{Y}(\mathfrak{gl}_{m|n})$ has a weight decomposition ($\mathbf{P}$-grading) with respect to Cartan subalgebra of $\mathrm{U}(\mathfrak{gl}_{m|n}) \subset \mathrm{Y}(\mathfrak{gl}_{m|n})$. The generator $t_{ij}^{(k)}$ has weight $\epsilon_i - \epsilon_j$.

We have the standard PBW theorem.

Theorem 2.4 ([Gow07])**.** *Fix some ordering on the generators $t_{ij}^{(k)}$, $i, j \in \bar{I}$ and $k \in \mathbb{Z}_{>0}$, for the super Yangian $\mathrm{Y}(\mathfrak{gl}_{m|n})$. Then the ordered monomials of these generators, with at most power 1 for odd generators, form a basis of $\mathrm{Y}(\mathfrak{gl}_{m|n})$.*

Set $\mathcal{B} := 1 + u^{-1}\mathbb{C}[[u^{-1}]]$. For any series $\vartheta(u) \in \mathcal{B}$, the map

$$\Gamma_\vartheta : T(u) \mapsto \vartheta(u)T(u) \tag{2.17}$$

defines an automorphism of $\mathrm{Y}(\mathfrak{gl}_{m|n})$. Denote by $\mathrm{Y}(\mathfrak{sl}_{m|n})$ the subalgebra of $\mathrm{Y}(\mathfrak{gl}_{m|n})$ which consists of all elements that are fixed under automorphisms Γ_ϑ for all $\vartheta(u) \in \mathcal{B}$.

Let $\mathfrak{z}_{m|n}$ be the center of super Yangian $\mathrm{Y}(\mathfrak{gl}_{m|n})$. If $m \neq n$, then we have an isomorphism of algebras

$$\mathrm{Y}(\mathfrak{gl}_{m|n}) \cong \mathfrak{z}_{m|n} \otimes \mathrm{Y}(\mathfrak{sl}_{m|n}).$$

see [Gow07, Proposition 8.1].

Let $\mathrm{Y}(\mathfrak{gl}_{n|m})$ be the super Yangian defined in the same way as $\mathrm{Y}(\mathfrak{gl}_{m|n})$ by interchanging m and n.

Let $\eta_{m|n}$ be the automorphism of $\mathrm{Y}(\mathfrak{gl}_{m|n})$ given by

$$\eta_{m|n} : T(u) \mapsto T(-u)^{-1}. \tag{2.18}$$

Define an isomorphism of superalgebras $\varrho_{m|n} : \mathrm{Y}(\mathfrak{gl}_{m|n}) \mapsto \mathrm{Y}(\mathfrak{gl}_{n|m})$ by

$$\varrho_{m|n} : t_{ij}(u) \mapsto t_{m+n+1-i,m+n+1-j}(-u).$$

Denote by $\hat{\varsigma}$ the composition of isomorphisms of superalgebras

$$\hat{\varsigma}_{m|n} = \varrho_{m|n} \circ \eta_{m|n}, \qquad \hat{\varsigma}_{m|n} : \mathrm{Y}(\mathfrak{gl}_{m|n}) \mapsto \mathrm{Y}(\mathfrak{gl}_{n|m}). \tag{2.19}$$

Finally, for fixed $m', n' \in \mathbb{Z}_{\geqslant 0}$, one also defines a larger super Yangian $\mathrm{Y}(\mathfrak{gl}_{m'+m|n'+n})$ following the choice of parities as in (2.3).

2.4. **Gauss decomposition.** The Gauss decomposition of $\mathrm{Y}(\mathfrak{gl}_{m|n})$, see [Gow07, Pen16], gives generating series

$$e_{ij}(u) = \sum_{r\geqslant 1} e_{ij}^{(r)} u^{-r}, \quad f_{ji}(u) = \sum_{r\geqslant 1} f_{ji}^{(r)} u^{-r}, \quad d_k(u) = 1 + \sum_{r\geqslant 1} d_k^{(r)} u^{-r},$$

where $1 \leqslant i < j \leqslant m+n$ and $k \in \bar{I}$, such that

$$t_{ii}(u) = d_i(u) + \sum_{k<i} f_{ik}(u) d_k(u) e_{ki}(u),$$

$$t_{ij}(u) = d_i(u) e_{ij}(u) + \sum_{k<i} f_{ik}(u) d_k(u) e_{kj}(u),$$

$$t_{ji}(u) = f_{ji}(u) d_i(u) + \sum_{k<i} f_{jk}(u) d_k(u) e_{ki}(u).$$

For $i \in I$ and $k \in \bar{I}$, let

$$e_i(u) = e_{i,i+1}(u) = \sum_{r\geqslant 1} e_i^{(r)} u^{-r}, \quad f_i(u) = f_{i+1,i}(u) = \sum_{r\geqslant 1} f_i^{(r)} u^{-r},$$

$$d'_k(u) = (d_k(u))^{-1} = 1 + \sum_{r\geqslant 1} d_k'^{(r)} u^{-r}.$$

We use the convention $d_k^{(0)} = d_k'^{(0)} = 1$.

The parities of $e_{ij}^{(r)}$ and $f_{ji}^{(r)}$ are the same as that of $t_{ij}^{(r)}$ while all $d_k^{(r)}$ and $d_k'^{(r)}$ are even. The super Yangian $\mathrm{Y}(\mathfrak{gl}_{m|n})$ is generated by $e_i^{(r)}$, $f_i^{(r)}$, $d_k^{(r)}$, $d_k'^{(r)}$, where $i \in I$ and $k \in \bar{I}$, and $r \geqslant 1$. The full defining relations are described in [Gow07, Lemma 4 or Theorem 3]. Here we only write down the following relations. Let $\phi_i(u) = d'_i(u)d_{i+1}(u) = 1 + \sum_{r\geqslant 1} \phi_i^{(r)} u^{-r}$. Then one has $[d_i^{(r)}, d_j^{(s)}] = 0$,

$$[d_i^{(r)}, e_j^{(s)}] = (\epsilon_i, \alpha_j) \sum_{t=0}^{r-1} d_i^{(t)} e_j^{(r+s-1-t)}, \quad [d_i^{(r)}, f_j^{(s)}] = -(\epsilon_i, \alpha_j) \sum_{t=0}^{r-1} f_j^{(r+s-1-t)} d_i^{(t)}, \tag{2.20}$$

$$[e_j^{(r)}, f_k^{(s)}] = -s_{j+1}\delta_{jk} \sum_{t=0}^{r+s-1} d_j'^{(t)} d_{j+1}^{(r+s-1-t)} = -s_{j+1}\delta_{jk}\phi_j^{(r+s-1)}. \tag{2.21}$$

Moreover, the subalgebra $\mathrm{Y}(\mathfrak{sl}_{m|n})$ is generated by the coefficients of the series $\phi_i(u)$, $e_i(u)$, $f_i(u)$ for $i \in I$.

Let $\mathrm{Y}^+_{m|n}$, $\mathrm{Y}^-_{m|n}$, and $\mathrm{Y}^0_{m|n}$ be the subalgebras of $\mathrm{Y}(\mathfrak{gl}_{m|n})$ generated by coefficients of the series $e_i(u)$, $f_i(u)$, and $d_j(u)$, respectively. It is known from [Gow07] that

$$\mathrm{Y}(\mathfrak{gl}_{m|n}) \cong \mathrm{Y}^-_{m|n} \otimes \mathrm{Y}^0_{m|n} \otimes \mathrm{Y}^+_{m|n}$$

as vector spaces and $d_i^{(r)}$ are algebraically free generators of $\mathrm{Y}^0_{m|n}$.

The Gauss decomposition for super Yangian associated to non-standard parity sequences is studied in [Pen16]. In particular, one obtains generating series $e_i(u)$, $f_i(u)$, $d_i(u)$, $d'_i(u)$ and generators $e_i^{(r)}$, $f_i^{(r)}$, $d_i^{(r)}$, $d_i'^{(r)}$ for $\mathrm{Y}(\mathfrak{gl}_{n|m})$ with standard parities and for $\mathrm{Y}(\mathfrak{gl}_{m'+m|n'+n})$ with parities in (2.3). We refer the reader to [Pen16, Tsy20] for the explicit relations of $\mathrm{Y}(\mathfrak{gl}_{m'+m|n'+n})$ in these generators.

We conclude this section with the following lemma used in Section 3.3.

Lemma 2.5 ([Gow07, Proposition 4.2]). *For the isomorphism $\hat{\varsigma}_{m|n} : \mathrm{Y}(\mathfrak{gl}_{m|n}) \mapsto \mathrm{Y}(\mathfrak{gl}_{n|m})$ defined in* (2.19)*, we have*

$$\hat{\varsigma}_{m|n} : d_i(u) \mapsto (d_{m+n+1-i}(u))^{-1},\ e_j(u) \mapsto -f_{m+n-j}(u),\ f_j(u) \mapsto -e_{m+n-j}(u),$$

for $i \in \bar{I}$ and $j \in I$. □

2.5. **Highest and lowest ℓ-weight representations.** Recall $\mathcal{B} = 1 + u^{-1}\mathbb{C}[[u^{-1}]]$ and set $\mathfrak{B} := \mathcal{B}^{\bar{I}} \times \mathbb{Z}_2$. We call an element $\boldsymbol{\zeta} \in \mathfrak{B}$ an *ℓ-weight*. We write ℓ-weights in the form $\boldsymbol{\zeta} = (\zeta_i(u))_{i\in\bar{I}}^{p(\boldsymbol{\zeta})}$, where $p(\boldsymbol{\zeta}) \in \mathbb{Z}_2$ and $\zeta_i(u) \in \mathcal{B}$ for all $i \in \bar{I}$.

Clearly $\mathfrak{B}$ is an abelian group with respect to the point-wise multiplication of the tuples and the addition of the parities. Let $\mathbb{Z}[\mathfrak{B}]$ be the group ring of $\mathfrak{B}$ whose elements are finite $\mathbb{Z}$-linear combinations of the form $\sum a_{\boldsymbol{\zeta}}[\boldsymbol{\zeta}]$, where $a_{\boldsymbol{\zeta}} \in \mathbb{Z}$.

Let M be a $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module. We say that a nonzero $\mathbb{Z}_2$-homogeneous vector $v \in M$ is *of ℓ-weight* $\boldsymbol{\zeta}$ if $d_i(u)v = \zeta_i(u)v$ for $i \in \bar{I}$ and the parity of v is given by $p(\boldsymbol{\zeta})$. We say that a vector $v \in M$ of ℓ-weight $\boldsymbol{\zeta}$ is a *highest (resp. lowest) ℓ-weight vector of ℓ-weight* $\boldsymbol{\zeta}$ if $e_{ij}(u)v = 0$ (resp. $f_{ji}(u)v = 0$) for all $1 \leqslant i < j \leqslant m+n$. The module M is called a *highest (resp. lowest) ℓ-weight module of ℓ-weight* $\boldsymbol{\zeta}$ if M is generated by a highest (resp. lowest) ℓ-weight vector of ℓ-weight $\boldsymbol{\zeta}$.

In general, ℓ-weight vectors do not need to be eigenvectors of $t_{ii}(u)$. However, from the Gauss decomposition one can deduce that v is a highest ℓ-weight vector of ℓ-weight $\boldsymbol{\zeta}$ if and only if v is of parity $p(\boldsymbol{\zeta})$ and

$$t_{ij}(u)v = 0, \quad t_{kk}(u)v = \zeta_k(u)v, \quad 1 \leqslant i < j \leqslant m+n,\ k \in \bar{I}. \tag{2.22}$$

Note similar formulas do not hold for a lowest ℓ-weight vector v of ℓ-weight $\boldsymbol{\zeta}$.

Let v and v' be highest ℓ-weight vectors of ℓ-weights $\boldsymbol{\zeta}$ and $\boldsymbol{\vartheta}$, respectively. Then, by (2.22) and (2.14), we have

$$t_{ij}(u)(v \otimes v') = 0, \quad t_{kk}(u)(v \otimes v') = \zeta_k(u)\vartheta_k(u)(v \otimes v'), \quad 1 \leqslant i < j \leqslant m+n,\ k \in \bar{I}.$$

Hence $v \otimes v'$ is a highest ℓ-weight vector of ℓ-weight $\boldsymbol{\zeta}\boldsymbol{\vartheta}$. In particular, we have

$$e_i(u)(v \otimes v') = 0, \quad d_j(u)(v \otimes v') = \zeta_j(u)\vartheta_j(u)(v \otimes v'), \quad i \in I,\ j \in \bar{I}. \tag{2.23}$$

This formula will be used to obtain information about $\Delta(d_j(u))$.

Every finite-dimensional irreducible $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module is a highest ℓ-weight module. Let $\boldsymbol{\zeta} \in \mathfrak{B}$ be an ℓ-weight. There exists a unique irreducible highest ℓ-weight $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module of highest ℓ-weight $\boldsymbol{\zeta}$. We denote it by $L(\boldsymbol{\zeta})$. The criterion for $L(\boldsymbol{\zeta})$ to be finite-dimensional is as follows.

Theorem 2.6 ([Zh96])**.** *The irreducible $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module $L(\boldsymbol{\zeta})$ is finite-dimensional if and only if there exist monic polynomials $g_i(u)$, $i \in \bar{I}$, such that*

$$\frac{\zeta_i(u)}{\zeta_{i+1}(u)} = \frac{g_i(u+s_i)}{g_i(u)}, \quad \frac{\zeta_m(u)}{\zeta_{m+1}(u)} = \frac{g_m(u)}{g_{m+n}(u)}, \quad i \in I,\ i \neq m,$$

and $\deg g_m = \deg g_{m+n}$. □

Finite-dimensional irreducible $\mathrm{Y}(\mathfrak{gl}_{m|n})$-modules stay irreducible under restriction to $\mathrm{Y}(\mathfrak{sl}_{m|n})$. Every irreducible finite-dimensional $\mathrm{Y}(\mathfrak{sl}_{m|n})$-module is a restriction of an irreducible finite-dimensional $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module. The restrictions of two finite-dimensional irreducible $\mathrm{Y}(\mathfrak{gl}_{m|n})$-modules are isomorphic $\mathrm{Y}(\mathfrak{sl}_{m|n})$-modules if and only if one of these modules is obtained from the other by a twist by the automorphism Γ_ϑ for $\vartheta(u) \in \mathcal{B}$.

We finish this section by proving the following technical proposition. Define the length function ℓ : $\mathbf{Q}_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0}$ by $\ell(\sum_{i\in I} n_i\alpha_i) = \sum_{i \in I} n_i$.

Proposition 2.7. *For $i \in I$, $j \in \bar{I}$, $k \in \mathbb{Z}_{>0}$, we have*

$$\Delta(d_j^{(k)}) - \sum_{l=0}^{k} d_j^{(l)} \otimes d_j^{(k-l)} \in \sum_{\ell(\alpha)>0} (\mathrm{Y}(\mathfrak{gl}_{m|n}))_\alpha \otimes (\mathrm{Y}(\mathfrak{gl}_{m|n}))_{-\alpha}, \tag{2.24}$$

$$\Delta(e_i^{(k)}) - 1 \otimes e_i^{(k)} \in \sum_{\ell(\alpha)>0} (\mathrm{Y}(\mathfrak{gl}_{m|n}))_\alpha \otimes (\mathrm{Y}(\mathfrak{gl}_{m|n}))_{\alpha_i-\alpha}, \tag{2.25}$$

$$\Delta(f_i^{(k)}) - f_i^{(k)} \otimes 1 \in \sum_{\ell(\alpha)>0} (\mathrm{Y}(\mathfrak{gl}_{m|n}))_{\alpha-\alpha_i} \otimes (\mathrm{Y}(\mathfrak{gl}_{m|n}))_{-\alpha}. \tag{2.26}$$

Proof. We simply write Y_α for $(\mathrm{Y}(\mathfrak{gl}_{m|n}))_\alpha$. Let N_i be the subalgebra of $\mathrm{Y}(\mathfrak{gl}_{m|n})$ generated by $e_j^{(r)}$ for $r \in \mathbb{Z}_{>0}$, $j \in I \setminus \{i\}$. Let A_i be the unital subalgebra of $\mathrm{Y}(\mathfrak{gl}_{m|n})$ generated by $\phi_i^{(r)}$, $r \in \mathbb{Z}_{>0}$. Let

$$h_i^{(1)} = d_i^{(2)} - \frac{1}{2}(d_i^{(1)})^2 - \frac{1}{2}d_i^{(1)},$$

then by (2.20), we have $[h_i^{(1)}, e_i^{(s)}] = c_i e_i^{(s+1)}$ for some $c_i \in \mathbb{C}^\times$. Note that $d_i^{(2)} = t_{ii}^{(2)} - \sum_{j<i} t_{ij}^{(1)} t_{ji}^{(1)}$ and $d_i^{(1)} = t_{ii}^{(1)}$. Direct computation implies that

$$\Delta(h_i^{(1)}) \in h_i^{(1)} \otimes 1 + 1 \otimes h_i^{(1)} + \mathbb{C}^\times e_{i-1}^{(1)} \otimes f_{i-1}^{(1)} + \mathbb{C}^\times e_i^{(1)} \otimes f_i^{(1)} + \sum_{\ell(\alpha)>1} (\mathrm{N}_i)_\alpha \otimes \mathrm{Y}_{-\alpha} + \sum_{\ell(\alpha-\alpha_i)>0} (\mathrm{N}_i)_\alpha \otimes \mathrm{Y}_{-\alpha}.$$

Note that $\Delta(e_i^{(1)}) = 1 \otimes e_i^{(1)} + e_i^{(1)} \otimes 1$. Using $[h_i^{(1)}, e_i^{(s)}] = c_i e_i^{(s+1)}$ and $[e_j^{(1)} \otimes f_j^{(1)}, 1 \otimes e_i^{(k)}] = 0$ for $j \neq i$ and $k \in \mathbb{Z}_{>0}$, one shows inductively that

$$\Delta(e_i^{(k)}) - 1 \otimes e_i^{(k)} \in \sum_{s=1}^{k} e_i^{(s)} \otimes A_i + \sum_{\ell(\alpha)>1} (\mathrm{N}_i)_\alpha \otimes \mathrm{Y}_{\alpha_i-\alpha} + \sum_{\ell(\alpha-\alpha_i)>0} (\mathrm{N}_i)_\alpha \otimes \mathrm{Y}_{\alpha_i-\alpha}.$$

In particular, we obtain (2.25). Similarly, one shows (2.26).

We then show (2.24). Since $\phi_i^{(k)} = -(\epsilon_{i+1}, \epsilon_{i+1})[e_i^{(k)}, f_i^{(1)}]$ and $f_i^{(1)}$ supercommutes with N_i, we have

$$\Delta(\phi_i^{(k)}) \in A_i \otimes A_i + \sum_{\ell(\alpha)>0} \mathrm{Y}_\alpha \otimes \mathrm{Y}_{-\alpha}.$$

Note that $d_1(u) = T_{11}(u)$, we have $\Delta(d_1(u)) = d_1(u) \otimes d_1(u) + \sum_{\ell(\alpha)>0} \mathrm{Y}_\alpha \otimes \mathrm{Y}_{-\alpha}[[u^{-1}]]$. Hence it suffices to show that

$$\Delta(\phi_i(u)) \in \phi_i(u) \otimes \phi_i(u) + \sum_{\ell(\alpha)>0} \mathrm{Y}_\alpha \otimes \mathrm{Y}_{-\alpha}[[u^{-1}]].$$

Let $\Delta_i(\phi_i^{(k)}) \in A_i \otimes A_i$ be such that $\Delta(\phi_i^{(k)}) - \Delta_i(\phi_i^{(k)}) \in \sum_{\ell(\alpha)>0} \mathrm{Y}_\alpha \otimes \mathrm{Y}_{-\alpha}$. Clearly, $A_i = \mathbb{C}[\phi_i^{(k)}]_{k>0}$ is a polynomial algebra, so is $A_i \otimes A_i$. Therefore an element $x \in A_i \otimes A_i$ is determined by the data $(\chi_1 \otimes \chi_2)(x)$ where χ_1, χ_2 are algebra homomorphisms $A_i \to \mathbb{C}$. Note that $\sum_{\ell(\alpha)>0} \mathrm{Y}_\alpha \otimes \mathrm{Y}_{-\alpha}$ annihilates tensor products of highest ℓ-weight vectors. Therefore, we have

$$(\chi_1 \otimes \chi_2)(\Delta_i(\phi_i^{(k)})) = \sum_{s=0}^{k} \chi_1(\phi_i^{(s)})\chi_2(\phi_i^{(k-s)}) = (\chi_1 \otimes \chi_2)\Big(\sum_{s=0}^{k} \phi_i^{(s)} \otimes \phi_i^{(k-s)}\Big),$$

where the first equality follows from (2.23). Therefore $\Delta_i(\phi_i^{(k)}) = \sum_{s=0}^{k} \phi_i^{(s)} \otimes \phi_i^{(k-s)}$, completing the proof of (2.24). □

2.6. **Evaluation maps.** The universal enveloping superalgebra $\mathrm{U}(\mathfrak{gl}_{m|n})$ is a Hopf subalgebra of $\mathrm{Y}(\mathfrak{gl}_{m|n})$ via the embedding $e_{ij} \mapsto s_i t_{ij}^{(1)}$. The left inverse of this embedding is the *evaluation homomorphism* $\pi_{m|n}$: $\mathrm{Y}(\mathfrak{gl}_{m|n}) \to \mathrm{U}(\mathfrak{gl}_{m|n})$ given by

$$\pi_{m|n} : t_{ij}(u) \mapsto \delta_{ij} + s_i e_{ij} u^{-1}. \tag{2.27}$$

The evaluation homomorphism is an algebra homomorphism but not a Hopf algebra homomorphism. For any $\mathfrak{gl}_{m|n}$-module M, it is naturally a $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module obtained by pulling back M through the evaluation homomorphism $\pi_{m|n}$. We denote the corresponding $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module by the same letter M and call it an *evaluation module*.

Following [Naz04], define the *modified evaluation map* of $\mathrm{Y}(\mathfrak{gl}_{m|n})$ by

$$\pi_{m|n}^{\iota} : t_{ij}(u) \mapsto \delta_{ij} + (-1)^{(|i|+1)(|j|+1)} e_{ji} u^{-1}.$$

Given a $\mathfrak{gl}_{m|n}$-module M, we call the $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module obtained by pulling back through the modified evaluation map $\pi_{m|n}^{\iota}$ a *modified evaluation module* and denote it by $\mathbb{M}$.

The evaluation map and modified one are related as follows. Let $\iota : \mathrm{Y}(\mathfrak{gl}_{m|n}) \to \mathrm{Y}(\mathfrak{gl}_{m|n})^{\mathrm{op}}$ be the isomorphism of Hopf superalgebras defined by

$$\iota : t_{ij}(u) \mapsto (-1)^{|i||j|+|i|} t_{ji}(-u), \tag{2.28}$$

where $\mathrm{Y}(\mathfrak{gl}_{m|n})^{\mathrm{op}}$ is the Hopf superalgebra with opposite coproduct (2.15). For a $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module M, denote by M^ι the pull back of M through ι.

Clearly, one has $\pi^\iota_{m|n} = \pi_{m|n} \circ \iota$. Therefore, the modified evaluation module can be thought of as the pull back of an evaluation module through the isomorphism ι, namely for a $\mathfrak{gl}_{m|n}$-module M, $\mathbb{M} = M^\iota$. Note that, more generally, for $z \in \mathbb{C}$, we have $\mathbb{M}_{-z} = (M^\iota)_z$.

Define also the *second modified evaluation map* $\pi^\vee_{m|n} : \mathrm{Y}(\mathfrak{gl}_{m|n}) \to \mathrm{U}(\mathfrak{gl}_{m|n})$ by

$$\pi^\vee_{m|n} = \varsigma_{n|m} \circ \pi_{n|m} \circ \hat{\varsigma}_{m|n},$$

where $\varsigma_{n|m}$, $\pi_{n|m}$, and $\hat{\varsigma}_{m|n}$ are defined in (2.2), (2.27), and (2.19), respectively. The second modified evaluation map will be used in Section 3.3.

Given a $\mathfrak{gl}_{m|n}$-module M, we call the $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module obtained by pulling back through the second modified evaluation map $\pi^\vee_{m|n}$ a *second modified evaluation module* and denote it by M.

Note that if $v \in M$ is a $\mathfrak{gl}_{m|n}$ singular vector of a given weight, then in the evaluation $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module M and second modified evaluation module M, v is a highest ℓ-weight vector, while in the modified evaluation module $\mathbb{M}$, v is a lowest ℓ-weight vector.

2.7. Category $\mathcal{C}$ and q-character map.

Let $\mathcal{C}$ be the category of finite-dimensional $\mathrm{Y}(\mathfrak{gl}_{m|n})$-modules. The category $\mathcal{C}$ is abelian and monoidal.

Let $M \in \mathcal{C}$ be a finite-dimensional $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module and $\boldsymbol{\zeta} \in \mathfrak{B}$ an ℓ-weight. Let

$$\zeta_i(u) = 1 + \sum_{j=1}^{\infty} \zeta_i^{(j)} u^{-j}, \qquad \zeta_i^{(j)} \in \mathbb{C}.$$

Denote by $M_{\boldsymbol{\zeta}}$ the *generalized ℓ-weight space* corresponding to the ℓ-weight $\boldsymbol{\zeta}$,

$$M_{\boldsymbol{\zeta}} := \{ v \in M \mid (d_i^{(j)} - \zeta_i^{(j)})^{\dim M} v = 0 \text{ for all } i \in \bar{I},\ j \in \mathbb{Z}_{>0}, \text{ and } |v| = p(\boldsymbol{\zeta}) \}.$$

For a finite-dimensional $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module M, define the *q-character* (or *Yangian character*) of M by the element

$$\chi(M) := \sum_{\boldsymbol{\zeta} \in \mathfrak{B}} \dim(M_{\boldsymbol{\zeta}})[\boldsymbol{\zeta}] \in \mathbb{Z}[\mathfrak{B}].$$

Let $\mathcal{R}ep(\mathcal{C})$ be the Grothendieck ring of $\mathcal{C}$, then χ induces a $\mathbb{Z}$-linear map from $\mathcal{R}ep(\mathcal{C})$ to $\mathbb{Z}[\mathfrak{B}]$.

Define the map $\varpi : \mathfrak{B} \to \mathfrak{h}^*$, $\boldsymbol{\zeta} \mapsto \varpi(\boldsymbol{\zeta})$ by $\varpi(\boldsymbol{\zeta})(e_{ii}) = s_i \zeta_i^{(1)}$.

Lemma 2.8. *The map $\chi : \mathcal{R}ep(\mathcal{C}) \to \mathbb{Z}[\mathfrak{B}]$ is an injective ring homomorphism.*

Proof. The fact that $\chi : \mathcal{R}ep(\mathcal{C}) \to \mathbb{Z}[\mathfrak{B}]$ is an ring homomorphism follows from Proposition 2.7, see e.g. [FR99, Remark 2.6]. Since $L(\boldsymbol{\zeta})$ is of highest ℓ-weight, by Theorem 2.4, $\chi(L(\boldsymbol{\zeta}))$ is equal to $[\boldsymbol{\zeta}]$ plus ℓ-weights of form $[\boldsymbol{\xi}]$ such that $\varpi(\boldsymbol{\xi})$ is strictly smaller than $\varpi(\boldsymbol{\zeta})$ with respect to the partial ordering on $\mathfrak{h}^*$. Therefore $[\boldsymbol{\zeta}]$ is the leading term in $\chi(L(\boldsymbol{\zeta}))$. Now the injectivity of χ is clear. □

In particular, we obtain the following.

Corollary 2.9. *The Grothendieck ring $\mathcal{R}ep(\mathcal{C})$ is commutative.* □

3. Skew representations and Jacobi-Trudi identity

3.1. **Skew representations.** Consider the embedding of $\mathfrak{gl}_{m'|n'}$ into $\mathfrak{gl}_{m'+m|n'+n}$ sending e_{ij} to e_{ij} for $i,j=1,2,\dots,m'+n'$. Here $\mathfrak{gl}_{m'|n'}$ has the standard parity and $\mathfrak{gl}_{m'+m|n'+n}$ has parity (2.3).

Let λ and μ be an $(m'+m|n'+n)$-hook partition and an $(m'|n')$-hook partition, respectively. Suppose further that $\lambda_i \geqslant \mu_i$ for all $i\in\mathbb{Z}_{>0}$. Consider the skew Young diagram λ/μ.

Let $\mu^\natural$ be the $\mathfrak{gl}_{m'|n'}$-weight corresponding to μ, see (2.7), and let λ° be the $\mathfrak{gl}_{m'+m|n'+n}$-weight corresponding to λ, see (2.9). We have the finite-dimensional irreducible $\mathfrak{gl}_{m'+m|n'+n}$-module $L(\lambda^\circ)$. Consider $L(\lambda^\circ)$ as a $\mathfrak{gl}_{m'|n'}$-module.

Define $L(\lambda/\mu)$ to be the subspace of $L(\lambda^\circ)$ by

$$L(\lambda/\mu):=\{v\in L(\lambda^\circ)\mid e_{ii}v=\mu^\natural(e_{ii})v,\ e_{jk}v=0,\ i=1,2,\dots,m'+n',\ 1\leqslant j<k\leqslant m'+n'\}.$$

Let $\varphi_{m'|n'}:\mathrm{Y}(\mathfrak{gl}_{m|n})\to\mathrm{Y}(\mathfrak{gl}_{m'+m|n'+n})$ be the embedding given by

$$\varphi_{m'|n'}:t_{ij}(u)\mapsto t_{m'+n'+i,m'+n'+j}(u).$$

Recall $\eta_{m|n}$ (and $\eta_{m'+m|n'+n}$) from (2.18). Let $\psi_{m'|n'}:\mathrm{Y}(\mathfrak{gl}_{m|n})\to\mathrm{Y}(\mathfrak{gl}_{m'+m|n'+n})$ be the injective homomorphism given by

$$\psi_{m'|n'}:=\eta_{m'+m|n'+n}\circ\varphi_{m'|n'}\circ\eta_{m|n}.$$

The following lemma can be found in [Pen16, Proof of Lemma 4.2].

Lemma 3.1 ([Gow07, Pen16]). *We have*

$$\psi_{m'|n'}(d_i(u))=d_{m'+n'+i}(u),\quad \psi_{m'|n'}(e_i(u))=e_{m'+n'+i}(u),\quad \psi_{m'|n'}(f_i(u))=f_{m'+n'+i}(u).\qquad\square$$

Regard $\mathrm{Y}(\mathfrak{gl}_{m'|n'})$ as the subalgebra of $\mathrm{Y}(\mathfrak{gl}_{m'+m|n'+n})$ via the natural embedding $t_{ij}(u)\mapsto t_{ij}(u)$ for $i,j=1,\dots,m'+n'$. We have the following lemma from [Pen16, Lemma 4.3].

Lemma 3.2 ([Pen16]). *The subalgebra $\mathrm{Y}(\mathfrak{gl}_{m'|n'})$ of $\mathrm{Y}(\mathfrak{gl}_{m'+m|n'+n})$ supercommutes with the image of $\mathrm{Y}(\mathfrak{gl}_{m|n})$ under the map $\psi_{m'|n'}$.* $\square$

Recall the evaluation map π, see (2.27), the following is straightforward from Lemma 3.2.

Corollary 3.3. *The image of the homomorphism*

$$\pi_{m'+m|n'+n}\circ\psi_{m'|n'}:\mathrm{Y}(\mathfrak{gl}_{m|n})\to\mathrm{U}(\mathfrak{gl}_{m'+m|n'+n})$$

supercommutes with the subalgebra $\mathrm{U}(\mathfrak{gl}_{m'|n'})$ in $\mathrm{U}(\mathfrak{gl}_{m'+m|n'+n})$. $\square$

Corollary 3.3 implies that the subspace $L(\lambda/\mu)$ is invariant under the action of the image of $\pi_{m'+m|n'+n}\circ\psi_{m'|n'}$. Therefore, $L(\lambda/\mu)$ is a $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module. We call $L(\lambda/\mu)$ a *skew representation*. We study the skew representations in the rest of this section.

3.2. **q-characters of skew representations.** In this section we compute the q-character of the $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module $L(\lambda/\mu)$.

Let $\kappa_i=i-1$ if $i=1,\dots,m$ and $\kappa_i=2m-i$ if $i=m+1,\dots,m+n$. Let

$$\mathfrak{X}_{i,a}=\big(1,\dots,\big(1+(u+a+\kappa_i)^{-1}\big)^{s_i},\dots,1\big)^{|i|}\in\mathfrak{B},\quad i\in\bar{I}.$$

Here the only component not equal to 1 is at the i-th position.

Recall that $\mathcal{T}(i,j)$ and $c(i,j)=j-i$ denote the number in the box representing the pair $(i,j)\in\lambda/\mu$ and the content of the pair (i,j) for a semi-standard Young tableau $\mathcal{T}$ of shape λ/μ, respectively.

It is known from [CPT15] that the dimension of $L(\lambda/\mu)$ is equal to the number of semi-standard Young tableaux of shape λ/μ. The following theorem is a refinement of this statement.

Theorem 3.4. *The q-character of the $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module $L(\lambda/\mu)$ is given by*

$$\mathcal{K}_{\lambda/\mu}(u) := \chi(L(\lambda/\mu)) = \sum_{\mathcal{T}} \prod_{(i,j)\in\lambda/\mu} \mathcal{X}_{\mathcal{T}(i,j),c(i,j)},$$

summed over all semi-standard Young tableaux $\mathcal{T}$ of shape λ/μ.

Before proving the theorem, we recall the following proposition from [Gow05, Theorems 1 and 2] and [Tsy20, Theorem 2.43].

Similar to κ_i, define κ_i', $i = 1,\dots,m'+n'$, with m and n replaced by m' and n', respectively. Set $\kappa'_{m'+n'+j} = m'-n'+\kappa_j$, $j\in\bar{I}$. Let $s_i' = (-1)^{|i|}$, $i = 1,2,\dots,m'+n'+m+n$, such that $|i|$ are chosen as in (2.3).

Proposition 3.5 ([Gow05, Tsy20]). *The coefficients of the series $\prod_{i\in\bar{I}}(d_i(u-\kappa_i))^{s_i}$ are central in $\mathrm{Y}(\mathfrak{gl}_{m|n})$. The coefficients of the series $\prod_{i=1}^{m'+n'+m+n}(d_i(u-\kappa_i'))^{s_i'}$ are central in $\mathrm{Y}(\mathfrak{gl}_{m'+m|n'+n})$.* □

Lemma 3.6. *Let λ be a Young diagram. Then the operator $\prod_{i\in\bar{I}}(d_i(u-\kappa_i))^{s_i}$ acts on the evaluation $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module $L(\lambda^\sharp)$ by the scalar operator*

$$\prod_{i\in\bar{I}}(d_i(u-\kappa_i))^{s_i}\Big|_{L(\lambda^\sharp)} = \prod_{i\in\bar{I}}\Big(1+\frac{(\lambda_i^\sharp,\epsilon_i)}{u-\kappa_i}\Big)^{s_i} = \prod_{(i,j)\in\lambda}\frac{u+c(i,j)+1}{u+c(i,j)}.$$

Similarly, the operator $\prod_{i=1}^{m'+n'+m+n}(d_i(u-\kappa_i'))^{s_i'}$ acts on the evaluation $\mathrm{Y}(\mathfrak{gl}_{m'+m|n'+n})$-module $L(\lambda^\circ)$ by the scalar operator

$$\prod_{i=1}^{m'+n'+m+n}(d_i(u-\kappa_i'))^{s_i'}\Big|_{L(\lambda^\circ)} = \prod_{(i,j)\in\lambda}\frac{u+c(i,j)+1}{u+c(i,j)}.$$

Proof. The statement follows from Proposition 3.5 and direct computations on highest ℓ-weight vector. □

Proof of Theorem 3.4. The proof is similar to [NT98, Lemma 2.1] and [FM02, Lemma 4.7]. Following the exposition of [Mol07, Corollary 8.5.8], Theorem 3.4 can be proved in a similar way using Lemma 3.1, Proposition 3.5, and Lemma 3.6. □

Remark 3.7. Due to Theorem 3.4, the q-character of $L(\lambda/\mu)$ relies only on the shape λ/μ and not on m', n'. Thus we have the module $L(\lambda/\mu)$ for arbitrary skew Young diagram λ/μ and its q-character is given by Theorem 3.4. Indeed, one can enlarge m' such that both λ and μ are hook partitions (of different indices). Moreover, we can also let $n'=0$, then the parity (2.3) is a standard parity sequence and the Lie superalgebra $\mathfrak{gl}_{m'+m|n'+n}$ is associated to a standard parity sequence. Therefore, one can always set $n'=0$ to simplify the discussion. □

For the $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module $L(\lambda/\mu)$, we write $L_z(\lambda/\mu)$ for $(L(\lambda/\mu))_z$. It is clear that

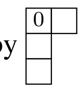
$$\chi(L_z(\lambda/\mu)) = \mathcal{K}_{\lambda/\mu}(u-z).$$

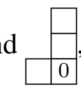
Remark 3.8. Recall that the pair $(i,j)\in\lambda/\mu$ is represented by the unit box whose south-eastern corner has coordinate $(i,j)\in\mathbb{Z}^2$. We can shift the whole Young diagram so that the numbers in the pair $(i,j)\in\mathbb{R}^2$ are not necessarily integers, but real numbers. The shape of the diagram remains the same. Only the contents of all boxes are shifted by the same number simultaneously. Hence we may consider the entire diagram is fixed and for any $z\in\mathbb{C}$ we can define the content c_z in a more general way, $c_z(i,j) = j-i-z$. Clearly, the q-character $\mathcal{K}_{\lambda/\mu}(u-z)$ is written in the same way as $\mathcal{K}_{\lambda/\mu}(u)$ in Theorem 3.4 by changing $c(i,j)$ to $c_z(i,j)$.

Note the contents of all boxes are uniquely determined by the content of a single box. Therefore, in the following, we shall sometimes specify the content of a box. If no content of a box is specified, then we are using the standard definition of content, namely $c(i,j) = j - i$. We also remark that for different Young diagrams $\lambda, \tilde{\lambda}, \mu, \tilde{\mu}$, the diagrams λ/μ and $\tilde{\lambda}/\tilde{\mu}$ may have the same shape but with possibly different contents. □

3.3. **Divisibility of q-characters.** In this section, we discuss the divisibility of q-characters of skew representations associated to special skew Young diagrams. We expect these observations would be helpful to understand Theorem 5.12 and prove Conjecture 5.14 below, see Section 5.5.

For a partition λ, denote by λ^- the skew Young diagram obtained by rotating λ by 180 degrees. By convention, we set the content of bottom-right box of λ^- to be zero.

Example 3.9. Let λ be the partition $(2,1,1)$, then λ and λ^- are given by 

and 

, respectively. Here the number 0 stands for the content of the corresponding box. □

Note that semi-standard Young tableaux of shape λ' are in one-to-one correspondence with semi-standard Young tableaux of shape λ^- given by changing numbers i in boxes of λ' to $m+n-i+1$ in the corresponding boxes of λ^-, see e.g. [Zha18, Lemma 2.7].

Let $\mathsf{L}_z(\lambda^\sharp)$ be the second modified evaluation module of $L(\lambda^\sharp)$ twisted by τ_z and set $\mathsf{L}(\lambda^\sharp) := \mathsf{L}_0(\lambda^\sharp)$.

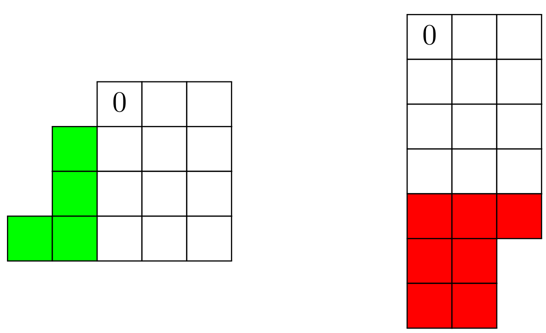
Lemma 3.10 ([Zha18, Theorem 2.4])**.** *We have the isomorphism of* $\mathrm{Y}(\mathfrak{gl}_{m|n})$*-modules,*

$$\mathsf{L}_z(\lambda^\sharp) \cong L_{m-n+z}(\lambda^-).$$

Proof. The lemma follows from Theorem 3.4 and Lemma 2.5, see also [Zha18, Theorem 2.4]. □

Let Ξ be the partition whose corresponding Young diagram is a rectangle of size $m \times n$. For a partition $\lambda \in \mathcal{P}(m|0)$, define $\mathbf{W}(\lambda)$ to be the skew Young diagram obtained by gluing Ξ and λ^- so that the bottom row of λ^- is next to the bottom row of the rectangular one exactly from left. Similarly, for $\mu \in \mathcal{P}(0|n)$, define $\mathbf{S}(\mu)$ to be the Young diagram obtained by attaching μ to the bottom of Ξ such that the first column of λ is exactly below the first column of Ξ. Moreover, we always assume that the box at left-upper corner of Ξ has content zero.

Example 3.11. Consider the case $m = 4$ and $n = 3$. Let $\lambda = (2,1,1)$ and $\mu = (3,2,2)$. Then the skew Young diagram $\mathbf{W}(\lambda)$ and the Young diagram $\mathbf{S}(\mu)$ are as follows.



Here the number zero means that the contents of the corresponding boxes are zero. We use the green color and red color to indicate the diagrams λ^- and μ corresponding to partitions λ and μ, respectively. □

Denote by χ_m the q-character map of $\mathrm{Y}(\mathfrak{gl}_m)$. We also have the evaluation modules for $\mathrm{Y}(\mathfrak{gl}_m)$, $\mathrm{Y}(\mathfrak{gl}_{0|n})$ defined by setting $n = 0$ and $m = 0$, respectively. Recall that $\mathcal{K}_{\lambda/\mu}(u) = \chi(L(\lambda/\mu))$, we use the superscripts to indicate the underlying algebra, e.g. $\mathcal{K}^{m|n}_{\lambda/\mu}(u)$, $\mathcal{K}^{m|0}_{\lambda/\mu}(u)$, and $\mathcal{K}^{0|n}_{\lambda/\mu}(u)$.

We identify an ℓ-weight $\boldsymbol{\zeta}_{[m]} = (\zeta_i(u))^{|0|}_{1\leqslant i\leqslant m}$ for $\mathrm{Y}(\mathfrak{gl}_m)$ with an ℓ-weight $\boldsymbol{\zeta} = (\zeta_i(u))^{|0|}_{i\in\bar{I}}$ for $\mathrm{Y}(\mathfrak{gl}_{m|n})$ via the natural embedding $\mathrm{Y}(\mathfrak{gl}_m) \hookrightarrow \mathrm{Y}(\mathfrak{gl}_{m|n})$, where $\zeta_j(u) = 1$ for $j = m+1, \dots, m+n$. Similarly, an ℓ-weight $\boldsymbol{\vartheta}_{[n]} = (\vartheta_i(u))^{p(\boldsymbol{\vartheta}_{[n]})}_{1\leqslant i\leqslant n}$ for $\mathrm{Y}(\mathfrak{gl}_{0|n})$ is identified with an ℓ-weight $\boldsymbol{\vartheta} = (\vartheta_i(u))^{p(\boldsymbol{\vartheta}_{[n]})}_{i\in\bar{I}}$ for $\mathrm{Y}(\mathfrak{gl}_{m|n})$ via the embedding $\psi_{m|0} : \mathrm{Y}(\mathfrak{gl}_{0|n}) \hookrightarrow \mathrm{Y}(\mathfrak{gl}_{m|n})$ in Lemma 3.1, where $\vartheta_j(u) = 1$ for $j = 1, \dots, m$.

Lemma 3.12. *We have the equalities of q-characters*

$$\mathcal{K}^{m|n}_{\mathbf{W}(\lambda)}(u) = \mathcal{K}^{m|0}_{\lambda^-}(u-m)\cdot \mathcal{K}^{m|n}_{\Xi}(u), \tag{3.1}$$

$$\mathcal{K}^{m|n}_{\mathbf{S}(\mu)}(u) = \mathcal{K}^{0|n}_{\mu}(u-m)\cdot \mathcal{K}^{m|n}_{\Xi}(u). \tag{3.2}$$

Proof. We first show (3.1). It is not hard to see that all semi-standard Young tableaux of shape $\mathbf{W}(\lambda)$ are obtained by independently filling numbers $1, \dots, m+n$ to the rectangle Ξ and numbers $1, \dots, m$ to λ^- so that each part gives a semi-standard Young tableau. In particular, $\mathcal{K}^{m|n}_{\mathbf{W}(\lambda)}(u)$ can be written as the product of $\mathcal{K}^{m|n}_{\Xi}(u)$ and the summation of monomials in $\mathcal{X}_{i,a}$ associated to semi-standard Young tableaux corresponding to λ^- filled by numbers $1, \dots, m$. Hence we obtain (3.1).

Similarly, all semi-standard Young tableaux of shape $\mathbf{S}(\mu)$ are obtained by independently filling numbers $1, \dots, m+n$ to the rectangle Ξ and numbers $m+1, \dots, m+n$ to μ so that each part gives a semi-standard Young tableau. The equality (3.2) is proved in a similar way. □

The lemma does not imply an equality on the representation level as not all q-characters are for $\mathrm{Y}(\mathfrak{gl}_{m|n})$.

Corollary 3.13. *We have the equality of q-characters*

$$\mathcal{K}^{m|n}_{\mathbf{W}(\lambda)}(u) = \chi_m(\mathsf{L}(\lambda^\sharp_{[m|0]}))\cdot \mathcal{K}^{m|n}_{\Xi}(u). \tag{3.3}$$

Proof. The statement follows from Lemma 3.10 and Lemma 3.12. □

Remark 3.14. Equations (3.1) and (3.2) are related by Lemma 3.10. Namely, ignoring the contents, we have the equality for skew Young diagrams

$$\big(\mathbf{W}_{n|m}(\mu')\big)' = \big(\mathbf{S}_{m|n}(\mu)\big)^-,$$

where $\mathbf{S}_{m|n}(\mu) := \mathbf{S}(\mu)$ is defined above while $\mathbf{W}_{n|m}(\mu')$ is defined in the same way as $\mathbf{W}_{m|n}(\lambda) := \mathbf{W}(\lambda)$ with m and n interchanged. □

Remark 3.15. Equation (3.2) implies that the $\mathfrak{gl}_{m|n}$-character of $L(\mathbf{S}(\mu)^\sharp)$ is equal to the $\mathfrak{gl}_{m|n}$-character of $L(\Xi^\sharp)$ multiplied by the $\mathfrak{gl}_n$-character of $L((\mu')^\sharp_{[n|0]})$. This fact can be understood observing that the $\mathfrak{gl}_{m|n}$-module $L(\mathbf{S}(\mu)^\sharp)$ is an irreducible Kac module. Indeed, let $\mathfrak{g}_{+1}$ and $\mathfrak{g}_{-1}$ be the odd subspaces of $\mathfrak{gl}_{m|n}$ spanned by $e_{i,m+j}$ and $e_{m+j,i}$ for all $i = 1, \dots, m$ and $j = 1, \dots, n$, respectively. Note the $\mathfrak{gl}_m$-module $L(\Xi^\sharp_{[m|0]})$ is one-dimensional. Extend the $\mathfrak{gl}_m \oplus \mathfrak{gl}_n$-module $L(\Xi^\sharp_{[m|0]}) \otimes L((\mu')^\sharp_{[n|0]})$ to the $\mathfrak{gl}_m \oplus \mathfrak{gl}_n \oplus \mathfrak{g}_{+1}$-module by putting $\mathfrak{g}_{+1}\big(L(\Xi^\sharp_{[m|0]}) \otimes L((\mu')^\sharp_{[n|0]})\big) = 0$, then we have the isomorphism of vector spaces by PBW theorem for $\mathfrak{gl}_{m|n}$,

$$L(\mathbf{S}(\mu)^\sharp) = \mathrm{Ind}^{\mathfrak{gl}_{m|n}}_{\mathfrak{gl}_m\oplus\mathfrak{gl}_n\oplus\mathfrak{g}_{+1}}\big(L(\Xi^\sharp_{[m|0]}) \otimes L((\mu')^\sharp_{[n|0]})\big) \cong L((\mu')^\sharp_{[n|0]}) \otimes \wedge^\bullet[\mathfrak{g}_{-1}],$$

where $\wedge^\bullet[\mathfrak{g}_{-1}]$ denotes the Grassmann algebra with mn variables and hence has dimension 2^{mn}. The part $\mathcal{K}^{0|n}_{\mu}(u-m)$ corresponds to $L((\mu')^\sharp_{[n|0]})$ while $\mathcal{K}^{m|n}_{\Xi}(u)$ corresponds to $\wedge^\bullet[\mathfrak{g}_{-1}]$.

Equations (3.1) and (3.3) can be interpreted similarly as well. □

3.4. **Jacobi-Trudi type identity.** Set $\mathcal{S}_k(u) = \mathcal{K}_{\lambda/\mu}(u)$ if $\lambda = (k)$ and $\mu = (0)$, and $\mathcal{A}_k(u) = \mathcal{K}_{\lambda/\mu}(u)$ if $\lambda = (1^k)$ and $\mu = (0)$.

We have the Jacobi-Trudi type identity for q-characters of skew representations.

Theorem 3.16. *We have*

$$\begin{aligned}\mathcal{K}_{\lambda/\mu}(u) &= \det_{1\leqslant i,j\leqslant \lambda_1'} \mathcal{S}_{\lambda_i-\mu_j-i+j}(u+\mu_j-j+1)\\ &= \det_{1\leqslant i,j\leqslant \lambda_1} \mathcal{A}_{\lambda_i'-\mu_j'-i+j}(u-\mu_j'+j-1).\end{aligned}$$

Here we use the convention that $\mathcal{S}_k(u) = \mathcal{A}_k(u) = 0$ *for* $k < 0$ *and* $\mathcal{S}_0(u) = \mathcal{A}_0(u) = 1$.

Proof. We give a proof of the first equality using the Lindström-Gessel-Viennot lemma, the second equality is proved similarly. We refer the reader to [Sag01, Chapter 4.5] for a detailed description of the method. Here we only give the necessary modifications to our situation. We remark that our adjustment is essentially the same as that of [Mol97, Theorem 3.1].

Consider the lattice plane $\mathbb{Z}^2$ (like the usual x-y coordinate plane and it should not be confused with the one defining Young diagrams) and lattice paths from one point to another. The paths consist of steps from one point to another of unit length northward or eastward or of length $\sqrt{2}$ northeastward. More precisely, the step starting from the point (i,j) can end at $(i+1,j)$ or $(i,j+1)$ if $1 \leqslant j \leqslant m$ and at $(i,j+1)$ or $(i+1,j+1)$ if $m < j \leqslant m+n$. We call an eastward or northeastward step a *contributed step*. For a contributed step $\mathfrak{s}$, denote by $\mathfrak{s}_x$ and $\mathfrak{s}_y$ the x-coordinate and y-coordinate of the starting point of $\mathfrak{s}$, respectively. For a lattice path $\mathfrak{p}$, define a monomial $\mathcal{X}^{\mathfrak{p}}$ in $\mathcal{X}_{i,a}$ by

$$\mathcal{X}^{\mathfrak{p}} = \prod_{\text{contributed } \mathfrak{s}\in\mathfrak{p}} \mathcal{X}_{\mathfrak{s}_y,\mathfrak{s}_x}.$$

Suppose the path $\mathfrak{p}$ is from the point (i_0,j_0) to the point (i_1,j_1), then it is clear that $\mathcal{X}^{\mathfrak{p}}$ is a term in $\mathcal{S}_{i_1-i_0}(u+i_0)$ as in Theorem 3.4. Moreover, we have

$$\mathcal{S}_{i_1-i_0}(u+i_0) = \sum_{\mathfrak{p}} \mathcal{X}^{\mathfrak{p}}$$

summed over all lattice paths $\mathfrak{p}$ from (i_0,j_0) to (i_1,j_1).

Set $l = \lambda_1'$. Let σ be an element of the symmetric group $\mathfrak{S}_l$ permuting numbers $1,\dots,l$. Let $\mathfrak{p}_i^\sigma$ be a lattice path from the point $(\mu_i-i+1,1)$ to the point $(\lambda_{\sigma(i)}-\sigma(i)+1, m+n+1)$ for $i=1,\dots,l$. Consider an l-tuple of paths $\boldsymbol{p}(\sigma) = (\mathfrak{p}_1^\sigma,\dots,\mathfrak{p}_l^\sigma)$. Define the monomial $\mathcal{X}^{\boldsymbol{p}(\sigma)}$ associated to $\boldsymbol{p}(\sigma)$ by

$$\mathcal{X}^{\boldsymbol{p}(\sigma)} = \prod_{i=1}^{l} \mathcal{X}^{\mathfrak{p}_i^\sigma}$$

if all $\mathfrak{p}_i^\sigma$ exist, otherwise set $\mathcal{X}^{\boldsymbol{p}(\sigma)} = 0$. Then one has

$$\det_{1\leqslant i,j\leqslant l} \mathcal{S}_{\lambda_i-\mu_j-i+j}(u+\mu_j-j+1) = \sum (-1)^{\mathrm{sign}(\sigma)} \mathcal{X}^{\boldsymbol{p}(\sigma)}, \tag{3.4}$$

where the summation is over all possible $\sigma \in \mathfrak{S}_l$ and all possible l-tuples $\boldsymbol{p}(\sigma)$.

We call a tuple $\boldsymbol{p}(\sigma)$ an *intersecting tuple* if $\mathfrak{p}_i^\sigma$ intersects $\mathfrak{p}_j^\sigma$ for some $i \neq j$. All monomials in (3.4) corresponding to intersecting tuples are cancelled out, see [Sag01, Chapter 4.5]. A tuple of the form $\boldsymbol{p}(\sigma)$ is non-intersecting only if $\sigma = \mathrm{id}$. Moreover, there exists a bijection between non-intersecting tuples with semi-standard Young tableaux of shape λ/μ. One direction of this bijection is described as follows, see an explicit example given below. Let $\boldsymbol{p}(\mathrm{id}) = (\mathfrak{p}_1^{\mathrm{id}},\dots,\mathfrak{p}_l^{\mathrm{id}})$ be a non-intersecting tuple, then filling the i-th row of the skew Young diagram λ/μ with the numbers $\mathfrak{s}_y$ for all contributed steps $\mathfrak{s}$ of $\mathfrak{p}_i^{\mathrm{id}}$ in non-decreasing order, for all $i = 1, \dots, l$, gives a semi-standard Young tableau of shape λ/μ. Note that $\mathfrak{s}_x$ corresponds to the content of the box in the i-th row filled with the number $\mathfrak{s}_y$. The theorem now follows from this bijection and Theorem 3.4. □

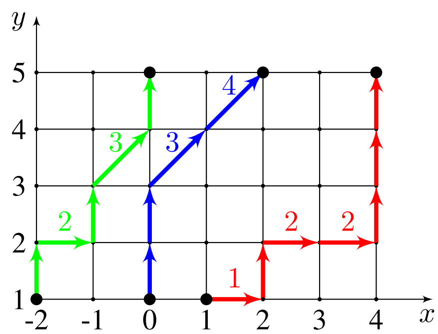
We exhibit an example explaining the correspondence between non-intersecting tuples and semi-standard Young tableaux.

Example 3.17. Let $\lambda = (4, 3, 2)$, $\mu = (1, 1, 0)$, $m = n = 2$. Consider the following semi-standard Young tableau.

$\mathcal{T} =$

	1	2	2
	3	4	
2	3		

In particular, the content of the box filled with number 1 is 1. Then it corresponds to the following 3-tuple of lattice paths.



Here we label each contributed step $\mathfrak{s}$ by the number $\mathfrak{s}_y$, the y-coordinate of the starting point of $\mathfrak{s}$. Then the labels from the first path (the rightmost path in red color) corresponds to the first row of $\mathcal{T}$. Similarly, the labels from the second (in blue color) and third (in green color) paths give rise to the numbers in the second and third rows of $\mathcal{T}$, respectively. The content of the box filled with number $\mathfrak{s}_y$ equals $\mathfrak{s}_x$, the x-coordinate of the starting point of $\mathfrak{s}$.

It is also very easy to write down the elements in the determinant $\det_{1\leqslant i,j\leqslant \lambda_1'} \mathcal{S}_{\lambda_i-\mu_j-i+j}(u+\mu_j-j+1)$ from the picture above. We order the starting points and end points from east to west. Then $\lambda_i - \mu_j - i + j$ corresponds to the horizontal length of any path from the j-th starting point to the i-th end point while $\mu_j - j + 1$ is the x-coordinate of the j-th starting point. Therefore, we have

$$\det_{1\leqslant i,j\leqslant \lambda_1'} \mathcal{S}_{\lambda_i-\mu_j-i+j}(u+\mu_j-j+1) = \begin{vmatrix} \mathcal{S}_3(u+1) & \mathcal{S}_4(u) & \mathcal{S}_6(u-2) \\ \mathcal{S}_1(u+1) & \mathcal{S}_2(u) & \mathcal{S}_4(u-2) \\ 0 & 1 & \mathcal{S}_2(u-2) \end{vmatrix},$$

where $0 = \mathcal{S}_{-1}(u+1)$ means there is no path from the first starting point to the third end point and $1 = \mathcal{S}_0(u)$ means there is exactly one path from the second starting point to the third end point. Moreover, this unique path contains no contributed steps. □

Clearly, it follows from Lemma 2.8 that there are corresponding identities on the level of representations in the Grothendieck ring $\mathcal{R}ep(\mathcal{C})$. Actually, Theorem 3.16 for the case of $\mathrm{Y}(\mathfrak{gl}_N)$ has been shown in [Che87, Che89] by resolutions of modules for the Yangian $\mathrm{Y}(\mathfrak{gl}_N)$. Ignoring the spectral parameter u, one obtains the Jacobi-Trudi identity for super-characters of Lie superalgebra $\mathfrak{gl}_{m|n}$, see [BB81].

Corollary 3.18. *If λ/μ contains a rectangle of size at least $(m+1)\times(n+1)$, then*

$$\det_{1\leqslant i,j\leqslant \lambda_1'} \mathcal{S}_{\lambda_i-\mu_j-i+j}(u+\mu_j-j+1)=\det_{1\leqslant i,j\leqslant \lambda_1} \mathcal{A}_{\lambda_i'-\mu_j'-i+j}(u-\mu_j'+j-1)=0.$$

Proof. If λ/μ contains a rectangle of size at least $(m+1)\times(n+1)$, then there are no semi-standard Young tableaux of shape λ/μ. Thus by Theorem 3.4 we have $\mathcal{K}_{\lambda/\mu}(u)=0$. The statement now follows from Theorem 3.16. □

4. Drinfeld functor and skew representations

4.1. **Degenerate affine Hecke algebras.** Let l be a positive integer. Following [Dri86], the *degenerate affine Hecke algebra* $\mathcal{H}_l$ is the associative algebra generated by generators $\sigma_1,\dots,\sigma_{l-1}$ and $x_1,\dots,x_l$ with the relations given by

$$\sigma_i^2=1,\quad \sigma_i\sigma_{i+1}\sigma_i=\sigma_{i+1}\sigma_i\sigma_{i+1},\quad [x_i,x_j]=0,$$

$$\sigma_i x_i = x_{i+1}\sigma_i-1,\quad [\sigma_j,\sigma_k]=[\sigma_j,x_k]=0\quad \text{if } |j-k|\neq 1.$$

As vector spaces, $\mathcal{H}_l\cong\mathbb{C}[\mathfrak{S}_l]\otimes\mathbb{C}[x_1,\dots,x_l]$. The generators $\sigma_1,\dots,\sigma_{l-1}$ generate a subalgebra isomorphic to $\mathbb{C}[\mathfrak{S}_l]$ while $x_1,\dots,x_l$ generate a subalgebra isomorphic to $\mathbb{C}[x_1,\dots,x_l]$. We shall use these identifications. It is well-known that the center of $\mathcal{H}_l$ is $\mathbb{C}[x_1,\dots,x_l]^{\mathfrak{S}_l}$.

Let σ_{ij} be the simple permutation (i,j). Let $y_1,\dots,y_l\in\mathcal{H}_l$ be defined by

$$y_1=x_1,\qquad y_i=x_i-\sum_{j<i}\sigma_{ji},\qquad i=2,\dots,l. \tag{4.1}$$

Then one has

$$\sigma y_i=y_{\sigma(i)}\sigma,\qquad [y_i,y_j]=-(y_i-y_j)\sigma_{ij},$$

for $\sigma\in\mathfrak{S}_l$ and $i,j=1,\dots,l$. Combining the relations in $\mathbb{C}[\mathfrak{S}_l]$, this could be considered as an alternate presentation of $\mathcal{H}_l$.

Consider the Lie algebra $\mathfrak{gl}_N$ and its Cartan part $\mathfrak{h}_N$. Let $\mathbf{\Phi}_N^+$ be the set of positive roots of $\mathfrak{gl}_N$ and $\mathbf{P}_N$ the weight lattice of $\mathfrak{gl}_N$. Let

$$\mathbf{D}_N^+:=\{\lambda\in\mathfrak{h}_N^*\mid (\lambda+\rho)(\alpha)\notin\mathbb{Z}_{<0}\text{ for all }\alpha\in\mathbf{\Phi}_N^+\},\qquad \mathbf{P}_N^+:=\mathbf{D}_N^+\cap\mathbf{P}_N.$$

We call an element of $\mathbf{D}_N^+$ (resp. $\mathbf{P}_N^+$) a *dominant* (resp. *dominant integral*) $\mathfrak{gl}_N$-weight. Recall that for a $\mathfrak{gl}_N$-module M, $\mathrm{wt}(M)$ denotes the set of all weights of M. In particular, we have

$$\mathrm{wt}((\mathbb{C}^N)^{\otimes l})=\Big\{\sum_{i=1}^N n_i\epsilon_i\mid n_i\in\mathbb{Z}_{\geqslant 0},\ \sum_{i=1}^N n_i=l\Big\}.$$

Now we recall some facts from representation theory of degenerate affine Hecke algebra $\mathcal{H}_l$, following [Zel80, Rog85].

Let $r\in\mathbb{Z}_{>0}$. Let $l=l_1+\dots+l_r$, where $l_i\in\mathbb{Z}_{\geqslant 0}$. Then algebra $\mathcal{H}_{l_1}\otimes\dots\otimes\mathcal{H}_{l_r}$ is identified to a subalgebra of $\mathcal{H}_l$ by the embedding

$$\mathcal{H}_{l_k}\hookrightarrow\mathcal{H}_l,\qquad \sigma_a\mapsto\sigma_{a+l_1+\dots+l_{k-1}},\qquad x_b\mapsto x_{b+l_1+\dots+l_{k-1}},$$

for $a=1,\dots,l_k-1$, $b=1,\dots,l_k$, and $k=1,\dots,r$.

Let M_i be an $\mathcal{H}_{l_i}$-module, $i=1,\dots,r$. Define the $\mathcal{H}_l$ module $M_1\odot\dots\odot M_r$ via the induction functor:

$$M_1\odot\dots\odot M_r:=\mathrm{Ind}_{\mathcal{H}_{l_1}\otimes\dots\otimes\mathcal{H}_{l_r}}^{\mathcal{H}_l}(M_1\otimes\dots\otimes M_r).$$

For complex numbers a, b such that $b-a+1=l$, denote by $\mathbb{C}_{[a,b]} := \mathbb{C}\mathbf{1}_{[a,b]}$ the one-dimensional representation of $\mathcal{H}_l$ given by

$$\sigma_i \cdot \mathbf{1}_{[a,b]} = \mathbf{1}_{[a,b]}, \qquad i=1,\dots,l-1, \tag{4.2}$$
$$x_j \cdot \mathbf{1}_{[a,b]} = (a+j-1)\mathbf{1}_{[a,b]}, \qquad j=1,2,\dots,l. \tag{4.3}$$

For $\lambda \in \mathfrak{h}_N^*$, set

$$\mathcal{W}(\lambda;l) := \Big\{\mu \in \mathfrak{h}_N^* \mid \lambda-\mu \in \mathrm{wt}((\mathbb{C}^N)^{\otimes l})\Big\}.$$

Take $\mu \in \mathcal{W}(\lambda;l)$ and set $\lambda_i=\lambda(\epsilon_i)$, $\mu_i=\mu(\epsilon_i)$, and $l_i=\lambda_i-\mu_i$, for $i=1,\dots,N$. Define an $\mathcal{H}_l$-module by

$$\mathcal{I}(\lambda,\mu) = \mathbb{C}_{[\mu_1,\lambda_1-1]} \odot \mathbb{C}_{[\mu_2-1,\lambda_2-2]} \odot \cdots \odot \mathbb{C}_{[\mu_N-N+1,\lambda_N-N]}. \tag{4.4}$$

We also set $\mathcal{I}(\lambda,\mu)=0$ if $\mu \notin \mathcal{W}(\lambda;l)$.

When λ is a dominant $\mathfrak{gl}_N$-weight and $\mu \in \mathcal{W}(\lambda;l)$, we call $\mathcal{I}(\lambda,\mu)$ a *standard module* of $\mathcal{H}_l$. The standard module can be thought as an analog of Verma modules.

Set $\mathbf{1}_{\lambda,\mu} := \mathbf{1}_{[\mu_1,\lambda_1-1]} \otimes \cdots \otimes \mathbf{1}_{[\mu_N-N+1,\lambda_N-N]}$, then there is an isomorphism of $\mathbb{C}[\mathfrak{S}_l]$-modules

$$\mathcal{I}(\lambda,\mu) \cong \mathbb{C}[\mathfrak{S}_l/\mathfrak{S}_{l_1}\times\cdots\times\mathfrak{S}_{l_N}]$$

induced by $\mathbf{1}_{\lambda,\mu} \mapsto 1$. In particular, one has the decomposition of $\mathfrak{S}_l$-modules,

$$\mathcal{I}(\lambda,\mu) \cong \mathcal{S}(\nu_{\lambda,\mu}) \oplus \bigoplus_{\nu>\nu_{\lambda,\mu}} \mathcal{S}(\nu)^{\oplus k_\nu},$$

where $\nu_{\lambda,\mu}$ is the partition obtained by rearranging the sequence $(l_1,\dots,l_N)$ in non-increasing order and $>$ denotes the dominance order of partitions. Here k_ν are non-negative integers.

It is well-known, see [Zel80, Rog85], that if $\lambda \in \mathbf{D}_N^+$, then $\mathcal{I}(\lambda,\mu)$ is generated by the subspace $\mathcal{S}(\nu_{\lambda,\mu})$ over $\mathcal{H}_l$. Therefore, $\mathcal{I}(\lambda,\mu)$ has a unique irreducible quotient $\mathcal{L}(\lambda,\mu)$ containing $\mathcal{S}(\nu_{\lambda,\mu})$. The $\mathfrak{S}_l$-module $\mathcal{S}(\nu_{\lambda,\mu})$ appears in $\mathcal{L}(\lambda,\mu)$ (considered as an $\mathfrak{S}_l$-module) with multiplicity one.

One has the following analog of the BGG resolution for $\mathcal{H}_l$-modules.

Proposition 4.1. [Che87] *Let $\lambda \in \mathfrak{h}_N^*$ and $\mu \in \mathcal{W}(\lambda;l)$. Suppose $\lambda-\rho \in \mathbf{P}_N^+$ and $\mu-\rho \in \mathbf{P}_N^+$, then there exists an exact sequence of $\mathcal{H}_l$-modules,*

$$0 \to \bigoplus_{\sigma\in\mathfrak{S}_N[N(N-1)/2]} \mathcal{I}(\lambda,\sigma\cdot\mu) \to \cdots \to \bigoplus_{\sigma\in\mathfrak{S}_N[1]} \mathcal{I}(\lambda,\sigma\cdot\mu) \to \mathcal{I}(\lambda,\mu) \to \mathcal{L}(\lambda,\mu) \to 0,$$

where $\mathfrak{S}_N[k]$ denotes the set of all elements of length k in $\mathfrak{S}_N$.

4.2. **Drinfeld functor.** In this section, we define the Drinfeld functor [Dri86] for super Yangian $\mathrm{Y}(\mathfrak{gl}_{m|n})$, following the exposition in [Ara99].

Let M be an $\mathcal{H}_l$-module. Consider the $\mathcal{H}_l \otimes \mathrm{U}(\mathfrak{gl}_{m|n})$-module $M \otimes V^{\otimes l}$, where $V=\mathbb{C}^{m|n} \cong L(\epsilon_1)$ is the vector representation of $\mathfrak{gl}_{m|n}$. For $i=1,\dots,l$, let

$$Q^{(i)} = (\mathcal{P}^{(0,i)})^{\top_0} = \sum_{a,b\in\bar{I}} (-1)^{|a||b|+|a|+|b|} E_{ab} \otimes E_{ab}^{(i)} \in \mathrm{End}(V) \otimes \mathrm{End}(V^{\otimes l}).$$

There is an algebra homomorphism

$$\begin{aligned}\wp:\ \mathrm{Y}(\mathfrak{gl}_{m|n}) &\to \mathcal{H}_l \otimes \mathrm{End}(V^{\otimes l}),\\ T(u) &\mapsto \mathfrak{T}_1(u-x_1)\mathfrak{T}_2(u-x_2)\cdots\mathfrak{T}_l(u-x_l),\end{aligned}$$

where

$$\mathfrak{T}_i(u-x_i) = 1 + \frac{1}{u-x_i} \otimes Q^{(i)}.$$

Thus $M\otimes V^{\otimes l}$ becomes a $\mathrm{Y}(\mathfrak{gl}_{m|n})$ module. One can think that it is a tensor product of evaluation $\mathrm{Y}(\mathfrak{gl}_{m|n})$-modules with value in M, where the i-th copy of V is evaluated at $x_i \in \mathcal{H}_l$, see (2.14) and (2.27).

The symmetric group acts naturally on $M \otimes V^{\otimes l}$ by $\sigma_i \mapsto \sigma_i \otimes \mathcal{P}^{(i,i+1)}$, $i = 1, \dots, l-1$, where $\mathcal{P}^{(i,i+1)} \in \mathrm{End}(V^{\otimes l})$ is defined in (2.8). Set

$$\mathcal{D}_l(M) := (M \otimes V^{\otimes l})/\sum_{i=1}^{l-1} \mathrm{Im}(\sigma_i + 1), \tag{4.5}$$

where $\mathrm{Im}(\sigma_i + 1)$ denotes the image of $\sigma_i + 1$ acting on $M \otimes V^{\otimes l}$.

Lemma 4.2. *The subspace $\sum_{i=1}^{l-1} \mathrm{Im}(\sigma_i + 1) \subset M \otimes V^{\otimes l}$ is an $\mathrm{Y}(\mathfrak{gl}_{m|n})$-submodule. In particular, $\mathcal{D}_l(M)$ is a $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module.*

Proof. The proof is parallel to [Ara99, Proposition 3]. We only show $[\mathcal{P}^{(i,i+1)}, Q^{(i)}] = [Q^{(i+1)}, Q^{(i)}]$. This is obtained from $[\mathcal{P}^{(i,i+1)}, \mathcal{P}^{(0,i)}] = [\mathcal{P}^{(0,i)}, \mathcal{P}^{(0,i+1)}]$ by applying the supertransposition $\top$ to the 0-th factor of $\mathrm{End}(V) \otimes \mathrm{End}(V^{\otimes l})$, see (2.5). □

Denote by $\mathcal{C}_{\mathcal{H}_l}$ the category of finite-dimensional representations of $\mathcal{H}_l$. Recall that $\mathcal{C}$ is the category of finite-dimensional representations of $\mathrm{Y}(\mathfrak{gl}_{m|n})$. The functor $\mathcal{D}_l$ is an exact functor from $\mathcal{C}_{\mathcal{H}_l}$ to $\mathcal{C}$. We call $\mathcal{D}_l$ the *Drinfeld functor*, cf. [Dri86].

In the case $n = 0$ we recover the standard Drinfeld functor. In that case, we have the following useful well-known theorem. We say that a representation of $\mathrm{Y}(\mathfrak{sl}_m)$ is *of level l* if all its irreducible components when restricted as $\mathfrak{sl}_m$-modules are submodules of $(\mathbb{C}^m)^{\otimes l}$. Denote by $\mathcal{C}_m^{(l)}$ the category of finite-dimensional representations of $\mathrm{Y}(\mathfrak{sl}_m)$ with level l.

Theorem 4.3 ([Dri86, CP96])**.** *If $l < m$, then the Drinfeld functor $\mathcal{D}_l$ is an equivalence between the category $\mathcal{C}_{\mathcal{H}_l}$ and the category $\mathcal{C}_m^{(l)}$.* □

A supersymmetric version of Theorem 4.3 for quantum affine superalgebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_{m|n})$ was recently proved in [Fli20] when $l < m + n$.

The following lemma describes the relation between M considered as an $\mathfrak{S}_l$-module and $\mathcal{D}_l(M)$ considered as a $\mathfrak{gl}_{m|n}$-module.

Lemma 4.4. *Let M be an $\mathcal{H}_l$-module. Let $M = \bigoplus_\nu \mathcal{S}(\nu')^{\oplus k_\nu}$ be the decomposition of M as an $\mathfrak{S}_l$-module, where the sum is over all partitions of l and $k_\nu \in \mathbb{Z}_{\geqslant 0}$. Then we have the decomposition of $\mathfrak{gl}_{m|n}$-modules,*

$$\mathcal{D}_l(M) \cong \bigoplus_{\nu \in \mathcal{P}_l(m|n)} L(\nu^\natural)^{\oplus k_\nu}.$$

Proof. The statement follows from Theorem 2.3 (Schur-Sergeev duality), see the proof of [Ara99, Proposition 4]. □

Lemma 4.5. *Let M_1 be an $\mathcal{H}_{l_1}$-module and M_2 an $\mathcal{H}_{l_2}$-module. Then we have the $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module isomorphism*

$$\mathcal{D}_{l_1}(M_1) \otimes \mathcal{D}_{l_1}(M_2) \cong \mathcal{D}_{l_1+l_2}(M_1 \odot M_2).$$

Proof. The proof is similar to that of [CP96, Proposition 4.7] or [Naz99, Proposition 5.3]. □

Lemma 4.6. *Let M be an $\mathcal{H}_l$-module. Then the action of $\mathrm{Y}(\mathfrak{gl}_{m|n})$ on $\mathcal{D}_l(M)$ can be written in the form*

$$t_{ij}(u) = \delta_{ij} + \sum_{k=1}^{l} \frac{1}{u - y_k} \otimes (-1)^{|i|} E_{ij}^{(k)},$$

where y_k are given by (4.1). Specifically, $t_{ij}^{(a)}$ acts by $\sum_{k=1}^{l} y_k^{a-1} \otimes (-1)^{|i|} E_{ij}^{(k)}$ for $a \geqslant 1$.

Proof. The proof is parallel to that of [Ara99, Proposition 6]. Note that $Q^{(i)} \cdot Q^{(k)} = \mathcal{P}^{(i,k)} \cdot Q^{(k)}$ follows from $\mathcal{P}^{(0,k)} \cdot \mathcal{P}^{(0,i)} = \mathcal{P}^{(i,k)} \cdot \mathcal{P}^{(0,k)}$ by applying supertransposition $\top$ to the 0-th factor of $\mathrm{End}(V) \otimes \mathrm{End}(V^{\otimes l})$. □

Define $\omega_k \in \mathfrak{h}^*$, $k \in \mathbb{Z}_{>0}$, by

$$\omega_k = \begin{cases} \epsilon_1 + \cdots + \epsilon_k, & \text{if } k \leqslant m; \\ \epsilon_1 + \cdots + \epsilon_m + (k-m)\epsilon_{m+1}, & \text{if } k \geqslant m. \end{cases}$$

Let $\lambda \in \mathfrak{h}_N^*$ and $\mu \in \mathcal{W}(\lambda; l)$. Set $\lambda_i = \lambda(\epsilon_i)$, $\mu_i = \mu(\epsilon_i)$, and $l_i = \lambda_i - \mu_i$, for $i = 1, \ldots, N$. Recall the $\mathcal{H}_l$-module $\mathcal{I}(\lambda, \mu)$ defined in (4.4).

Lemma 4.7. *Let a, b be complex numbers such that $b - a + 1 = l$. Then we have the $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module isomorphism $\mathcal{D}_l(\mathbb{C}_{[a,b]}) \cong L_a(\omega_l)$. Moreover, we have the $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module isomorphism*

$$\mathcal{M}(\lambda, \mu) := \mathcal{D}_l(\mathcal{I}(\lambda, \mu)) \cong L_{\mu_1}(\omega_{l_1}) \otimes L_{\mu_2 - 1}(\omega_{l_2}) \otimes \cdots \otimes L_{\mu_N - N + 1}(\omega_{l_N}).$$

Proof. From (4.1), (4.2), and (4.3) we obtain that $y_i \cdot \mathbf{1}_{[a,b]} = a\mathbf{1}_{[a,b]}$. Therefore, the first statement follows from Lemma 4.4 and Lemma 4.6. The second statement follows from the first statement and Lemma 4.5. □

Unlike the $\mathfrak{gl}_N$ case [Ara99, Theorem 9], when $mn > 0$, the $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module $\mathcal{M}(\lambda, \mu)$ is never zero.

4.3. **Drinfeld functor and skew representations.** In this section, we study the relations between skew representations and Drinfeld functor. We need the following proposition.

Proposition 4.8. *Let M be a finite-dimensional irreducible representation of $\mathcal{H}_l$, then the $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module $\mathcal{D}_l(M)$ is irreducible.*

Proof. The proof is similar to that of [Naz99, Theorem 5.5]. □

We investigate the $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module $\mathcal{D}_l(\mathcal{L}(\lambda, \mu))$. Note that the highest ℓ-weight vector of $\mathcal{D}_l(\mathcal{L}(\lambda, \mu))$ is not given by the quotient image of tensor product of highest ℓ-weight vectors of all $L_{\mu_i - i + 1}(\omega_{l_i})$ in general and therefore the computation of highest ℓ-weight of $\mathcal{D}_l(\mathcal{L}(\lambda, \mu))$ is not straightforward. We use the resolution of the $\mathcal{H}_l$-module $\mathcal{L}(\lambda, \mu)$ and Jacobi-Trudi identity of q-characters to show that $\mathcal{D}_l(\mathcal{L}(\lambda, \mu))$ is a skew representation if $\lambda_i - \lambda_{i+1} \in \mathbb{Z}_{\geqslant 0}$ and $\mu_i - \mu_{i+1} \in \mathbb{Z}_{\geqslant 0}$ for all $i = 1, \ldots, N-1$.

For $\lambda \in \mathfrak{h}_N^*$ such that $\lambda - \rho \in \mathbf{P}_N^+$, we identify the weight λ with a partition in the usual way. We denote this partition also by λ.

Theorem 4.9. *Let $\lambda \in \mathfrak{h}_N^*$ and $\mu \in \mathcal{W}(\lambda; l)$. Suppose $\lambda - \rho \in \mathbf{P}_N^+$ and $\mu - \rho \in \mathbf{P}_N^+$, then we have $\mathcal{D}_l(\mathcal{L}(\lambda, \mu)) \cong L(\lambda'/\mu')$ as $\mathrm{Y}(\mathfrak{gl}_{m|n})$-modules. In particular, the skew representation $L(\lambda'/\mu')$ is irreducible.*

Proof. Applying the Drinfeld functor $\mathcal{D}_l$ to the resolution of the $\mathcal{H}_l$-module $\mathcal{L}(\lambda, \mu)$ in Proposition 4.1, we have the exact sequence of $\mathrm{Y}(\mathfrak{gl}_{m|n})$-modules,

$$0 \to \bigoplus_{\sigma \in \mathfrak{S}_N[N(N-1)/2]} \mathcal{M}(\lambda, \sigma \cdot \mu) \to \cdots \to \bigoplus_{\sigma \in \mathfrak{S}_N[1]} \mathcal{M}(\lambda, \sigma \cdot \mu) \to \mathcal{M}(\lambda, \mu) \to \mathcal{D}_l(\mathcal{L}(\lambda, \mu)) \to 0.$$

For $\sigma \in \mathfrak{S}_N$, it is clear from Lemma 2.8 and Lemma 4.7 that

$$\chi(\mathcal{M}(\lambda, \sigma \cdot \mu)) = \prod_{i=1}^{N} \mathcal{A}_{\lambda_i - \mu_{\sigma^{-1}(i)} - i + \sigma^{-1}(i)}(u - \mu_{\sigma^{-1}(i)} + \sigma^{-1}(i) - 1),$$

where $N \geqslant \lambda_1'$. By Theorem 3.16 and the resolution above, we obtain that

$$\chi(\mathcal{D}_l(\mathcal{L}(\lambda, \mu))) = \mathcal{K}_{\lambda'/\mu'}(u) = \chi(L(\lambda'/\mu')).$$

Since by Proposition 4.8, $\mathcal{D}_l(\mathcal{L}(\lambda,\mu))$ is an irreducible $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module, we conclude again from Lemma 2.8 that $\mathcal{D}_l(\mathcal{L}(\lambda,\mu)) \cong L(\lambda'/\mu')$. In particular, $L(\lambda'/\mu')$ is irreducible. □

Give a partition λ of length at most N and a complex number z. Define $\mathfrak{gl}_N$-weights λ_z and 0_z by

$$\lambda_z(\epsilon_i) = \lambda_i + z, \qquad 0_z(\epsilon_i) = z, \qquad i = 1, \dots, N.$$

Corollary 4.10. *Suppose the same conditions as in Theorem 4.9 hold. Let z be an arbitrary complex number, then we have $\mathcal{D}_l(\mathcal{L}(\lambda_z,\mu_z)) \cong L_z(\lambda'/\mu')$.* □

4.4. **Fusion procedure and skew representations.** In this section, we study further skew representations by fusion procedure, following [Che86, NT02, Naz04].

Fix partitions λ and μ such that $\mu \subset \lambda$. Set $l = |\lambda| - |\mu|$. Let Ω be a standard Young tableau of shape λ/μ. For $i \in \{1,\dots,l\}$, denote the content of the box in Ω containing i by $c_i(\Omega)$. Consider the operator

$$\mathcal{E}_\Omega := \prod_{1\leqslant i<j\leqslant l}^{\longrightarrow} R_{ij}\big(c_i(\Omega) - c_j(\Omega)\big) \in \mathrm{End}(V^{\otimes l}),$$

where $V = \mathbb{C}^{m|n}$, $R_{ij}(u) = 1 - \mathcal{P}^{(i,j)}/u$ is the rational R-matrix for $\mathrm{Y}(\mathfrak{gl}_{m|n})$, and the order of the product corresponds to the writing of permutation $\sigma \in \mathfrak{S}_l$, $\sigma(i) = l+1-i$, in terms of simple transpositions.

It is known from [Che86] that $\mathcal{E}_\Omega$ is well-defined, for a proof see e.g. [NT02, Proposition 2.2].

Consider the tensor product of evaluation $\mathrm{Y}(\mathfrak{gl}_{m|n})$-modules $V_{z_1}\widetilde{\otimes}\cdots\widetilde{\otimes}V_{z_l}$ for $z_1,\dots,z_l \in \mathbb{C}$, where $\widetilde{\otimes}$ denotes the tensor product induced by the opposite coproduct (2.15). It follows from (2.15) and (2.27) that the corresponding homomorphism $\pi_{z_1,\dots,z_l} : \mathrm{Y}(\mathfrak{gl}_{m|n}) \to \mathrm{End}(V^{\otimes l})$ is given by

$$T(u) \mapsto R_{0,l}^{\top}(-u+z_l)\cdots R_{0,1}^{\top}(-u+z_1)$$

under the homomorphism

$$\mathrm{id}\otimes\pi_{z_1,\dots,z_l} : \mathrm{End}(V)\otimes\mathrm{Y}(\mathfrak{gl}_{m|n}) \to \mathrm{End}(V)\otimes\mathrm{End}(V^{\otimes l}),$$

where $\top$ stands for the supertranspose (2.5) on the 0-th factor of $\mathrm{End}(V)$ in $\mathrm{End}(V)\otimes\mathrm{End}(V^{\otimes l})$. Denote by $\mathfrak{P}$ the linear operator on $V^{\otimes l}$ reversing the order of the tensor factors.

Proposition 4.11. *Set $z_i = -c_i(\Omega)$ for $i = 1,\dots,l$. Then the operator $\mathcal{E}_\Omega \circ \mathfrak{P}$ is a $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module homomorphism*

$$\mathcal{E}_\Omega\circ\mathfrak{P} : V_{z_l}\widetilde{\otimes}\cdots\widetilde{\otimes}V_{z_1} \to V_{z_1}\widetilde{\otimes}\cdots\widetilde{\otimes}V_{z_l}.$$

In particular, the image of $\mathcal{E}_\Omega$ in $V^{\otimes l}$ is a submodule of the $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module $V_{z_1}\widetilde{\otimes}\cdots\widetilde{\otimes}V_{z_l}$.

Proof. The proof is similar to that of [Naz04, Proposition 4.2]. Note that our z_i corresponds to $-z_i$ there. Explicitly, the proof is modified by applying the supertransposition $\top$ on the 0-th copy of $\mathrm{End}(V)$ in $\mathrm{End}(V)\otimes\mathrm{End}(V)^{\otimes l}$ and replacing x with $-u$, cf. [Mol07, Section 6.5]. □

We denote by $\mathcal{F}(\Omega)$ the $\mathrm{Y}(\mathfrak{gl}_{m|n})$-submodule of $V_{z_1}\widetilde{\otimes}\cdots\widetilde{\otimes}V_{z_l}$ defined by the image of $\mathcal{E}_\Omega$ acting on $V^{\otimes l}$. Now we are ready to compare $\mathcal{F}(\Omega)$ with $L(\lambda/\mu)$.

Define the rational function $g_\mu(u)$ by

$$g_\mu(u) = \prod_{i\geqslant 1}\frac{(u+\mu_i-i)(u-i+1)}{(u+\mu_i-i+1)(u-i)}.$$

Then $g_\mu(\infty) = 1$ and we identify $g_\mu(u)$ as a series in $\mathcal{B} = 1 + u^{-1}\mathbb{C}[[u^{-1}]]$. Recall the automorphism defined by $\Gamma_\vartheta : T(u) \mapsto \vartheta(u)T(u)$ from (2.17) for $\vartheta(u) \in \mathcal{B}$. Denote by $\mathbb{C}_\vartheta$ the even one-dimensional $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module defined by the homomorphism

$$\varepsilon\circ\Gamma_{g_\mu} : \mathrm{Y}(\mathfrak{gl}_{m|n}) \to \mathrm{End}(\mathbb{C}_\vartheta), \qquad T(u) \mapsto \vartheta(u),$$

where ε is the counit map, see (2.14).

Theorem 4.12. *The $\mathrm{Y}(\mathfrak{gl}_{m|n})$-modules $L(\lambda/\mu)\otimes\mathbb{C}_{g_\mu}$ and $\mathcal{F}(\Omega)$ are isomorphic.*

Proof. The proof is similar to that of [Naz04, Theorem 1.6]. We only remark that the irreducibility of $L(\lambda/\mu)$ there is obtained from Olshanski's centralizer construction [MO00] of $\mathrm{Y}(\mathfrak{gl}_N)$. In this paper we obtain the irreducibility of $\mathrm{Y}(\mathfrak{gl}_{m|n})$ module $L(\lambda/\mu)$ in a different way using Drinfeld functor, see Theorem 4.9. □

We have the so-called binary property for tensor products of skew representations of $\mathrm{Y}(\mathfrak{gl}_{m|n})$, cf. [NT02, Theorem 4.9].

Theorem 4.13. *Let $\lambda^{(i)}$ and $\mu^{(i)}$ be partitions such that $\mu^{(i)}\subset\lambda^{(i)}$, $i=1,\dots,k$. Let $z_1,\dots,z_k$ be complex numbers. Then the $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module $L_{z_1}(\lambda^{(1)}/\mu^{(1)})\otimes\cdots\otimes L_{z_k}(\lambda^{(k)}/\mu^{(k)})$ is irreducible if and only if $L_{z_i}(\lambda^{(i)}/\mu^{(i)})\otimes L_{z_j}(\lambda^{(j)}/\mu^{(j)})$ is irreducible for all $1\leqslant i<j\leqslant k$.*

Proof. Let Ω_i be the column tableau of shape λ_i/μ_i for $i=1,\dots,k$. Denote by $\mathcal{F}_{z_i}(\Omega_i)$ the pull back of $\mathcal{F}(\Omega_i)$ through τ_{z_i}. Thanks to Theorem 4.12, it suffices to show that

$$\mathcal{F}_{z_1}(\Omega_1)\otimes\cdots\otimes\mathcal{F}_{z_k}(\Omega_k)$$

is irreducible if and only if $\mathcal{F}_{z_i}(\Omega_i)\otimes\mathcal{F}_{z_j}(\Omega_j)$ for all $1\leqslant i<j\leqslant k$, see [NT02, Theorems 4.8 and 4.9]. The argument in [NT02] using fusion procedure concerns the operators in the group algebra $\mathbb{C}[\mathfrak{S}_l]$ which can be generalized to the super setting with very few changes. Therefore the statement follows. □

For any $\mathrm{Y}(\mathfrak{gl}_{m|n})$-modules $M_1,\dots,M_k$, we have

$$(M_1\widetilde{\otimes}\cdots\widetilde{\otimes}M_k)^\iota\cong M_1^\iota\otimes\cdots\otimes M_k^\iota.$$

Let $g_\mu^\iota(u)=g_\mu(-u)$.

Theorem 4.14. *The $\mathrm{Y}(\mathfrak{gl}_{m|n})$-modules $L(\lambda/\mu)^\iota\otimes\mathbb{C}_{g_\mu^\iota}$ and $\mathcal{F}(\Omega)^\iota$ are isomorphic and the $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module $\mathcal{F}(\Omega)^\iota$ is a submodule of $\mathbb{V}_{c_1(\Omega)}\otimes\cdots\otimes\mathbb{V}_{c_l(\Omega)}$, where $\mathbb{V}$ is the modified evaluation vector representation.*

Proof. The statement follows from Proposition 4.11 and Theorem 4.12. □

4.5. **Application: irreducibility of tensor products.** Due to Theorem 4.3 and Proposition 4.8, many results from the representation theory of $\mathrm{Y}(\mathfrak{gl}_N)$ can be generalized to the case of $\mathrm{Y}(\mathfrak{gl}_{m|n})$. We give two such examples in this and next sections.

The following statement should be well-known for experts. However, we are not able to find the suitable reference, cf. [Zel80].

Proposition 4.15. *Let $\lambda^{(i)}$ and $\mu^{(i)}$ be partitions such that $\mu^{(i)}\subset\lambda^{(i)}$, $i=1,\dots,k$. Let $z_1,\dots,z_k$ be complex numbers such that $z_i-z_j\notin\mathbb{Z}$ for all $1\leqslant i<j\leqslant k$. Then the induction product*

$$\mathcal{L}(\lambda^{(1)}_{z_1},\mu^{(1)}_{z_1})\odot\cdots\odot\mathcal{L}(\lambda^{(k)}_{z_k},\mu^{(k)}_{z_k})$$

is an irreducible $\mathcal{H}_l$-module, where $l=\sum_{i=1}^k(|\lambda^{(i)}|-|\mu^{(i)}|)$.

Proof. Let N be sufficiently large. Applying the Drinfeld functor to $\mathcal{L}(\lambda^{(1)}_{z_1},\mu^{(1)}_{z_1})\odot\cdots\odot\mathcal{L}(\lambda^{(k)}_{z_k},\mu^{(k)}_{z_k})$ with $m=N$ and $n=0$, we obtain the $\mathrm{Y}(\mathfrak{gl}_N)$-module

$$L_{z_1}(\lambda^{(1)\prime}/\mu^{(1)\prime})\otimes\cdots\otimes L_{z_k}(\lambda^{(k)\prime}/\mu^{(k)\prime}),$$

which is known to be irreducible when $z_i-z_j\notin\mathbb{Z}$ for all $1\leqslant i<j\leqslant k$, see [NT98, Corollary 3.9]. The proposition follows from Theorem 4.3. □

The following theorem is a direct corollary of Proposition 4.8, Proposition 4.15, and Corollary 4.10.

Theorem 4.16. *Let $\lambda^{(i)}$ and $\mu^{(i)}$ be partitions such that $\mu^{(i)} \subset \lambda^{(i)}$, $i = 1, \dots, k$. Let $z_1, \dots, z_k$ be complex numbers such that $z_i - z_j \notin \mathbb{Z}$ for all $1 \leqslant i < j \leqslant k$. Then the tensor product of skew representations*

$$L_{z_1}(\lambda^{(1)}/\mu^{(1)}) \otimes \cdots \otimes L_{z_k}(\lambda^{(k)}/\mu^{(k)})$$

is an irreducible $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module. □

Let λ and μ be two partitions. Let N be sufficiently large. Define the numbers

$$a_i = \lambda_i - i + 1, \qquad b_i = \mu_i - i + 1, \qquad i = 1, \dots, N.$$

For each pair (i, j) such that $1 \leqslant i < j \leqslant N$, define the subsets of $\mathbb{Z}$ by

$$\langle a_j, a_i \rangle = \{a_j, a_j + 1, \dots, a_i\} \setminus \{a_j, a_{j+1}, \dots, a_i\},$$

$$\langle b_j, b_i \rangle = \{b_j, b_j + 1, \dots, b_i\} \setminus \{b_j, b_{j+1}, \dots, b_i\}.$$

Note that if $\lambda_i = \lambda_{i-1} = \cdots = \lambda_j$, then $\langle a_j, a_i \rangle = \varnothing$.

Proposition 4.17. *Let λ and μ be two partitions, z and w two complex numbers. Then the induction product $\mathcal{L}((\lambda')_z, 0_z) \odot \mathcal{L}((\mu')_w, 0_w)$ is irreducible if and only if for each pair (i, j) such that $1 \leqslant i < j \leqslant N$, we have*

$$b_j + z - w,\ b_i + z - w \notin \langle a_j, a_i \rangle \qquad \text{or} \qquad a_j - z + w,\ a_i - z + w \notin \langle b_j, b_i \rangle. \tag{4.6}$$

In particular, $\mathcal{L}((\lambda')_z, 0_z) \odot \mathcal{L}((\lambda')_z, 0_z)$ is irreducible.

Proof. The proof is similar to that of Proposition 4.15 using [Mol02, Theorem 1.1]. □

Theorem 4.18. *Let λ and μ be two partitions, z and w two complex numbers. Suppose the condition (4.6) holds for all pairs (i, j) such that $1 \leqslant i < j$, then the $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module $L_z(\lambda^\natural) \otimes L_w(\mu^\natural)$ is irreducible. In particular, $L_z(\lambda^\natural) \otimes L_z(\lambda^\natural)$ is irreducible.*

Proof. The theorem follows from Proposition 4.8 and Proposition 4.17. □

Combining Theorem 4.18 with Theorem 4.13, one is able to give sufficient conditions for a tensor product of evaluation $\mathrm{Y}(\mathfrak{gl}_{m|n})$-modules to be irreducible.

Comparing to the $\mathrm{Y}(\mathfrak{gl}_N)$ case, conditions (4.6) are not necessary for $L_z(\lambda^\natural) \otimes L_w(\mu^\natural)$ to be irreducible. It would be interesting to generalize [Mol02, Theorem 1.1] to skew representations of $\mathrm{Y}(\mathfrak{gl}_{m|n})$.

Example 4.19. We compare the sufficient and necessary conditions for $L_z(2\epsilon_1) \otimes L_w(2\epsilon_1)$ to be irreducible over $\mathrm{Y}(\mathfrak{gl}_2)$ and $\mathrm{Y}(\mathfrak{gl}_{1|1})$. The $\mathrm{Y}(\mathfrak{gl}_2)$-module $L_z(2\epsilon_1) \otimes L_w(2\epsilon_1)$ is irreducible if and only if $z - w \neq \pm 1, \pm 2$, while the $\mathrm{Y}(\mathfrak{gl}_{1|1})$-module $L_z(2\epsilon_1) \otimes L_w(2\epsilon_1)$ is irreducible if and only if $z - w \neq \pm 2$. Therefore the conditions (4.6) are not necessary for the irreducibility of the $\mathrm{Y}(\mathfrak{gl}_{1|1})$-module $L_z(2\epsilon_1) \otimes L_w(2\epsilon_1)$. □

We call an irreducible $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module M *real* if $M \otimes M$ is also irreducible, see [Lec03]. Theorem 4.18 implies that the evaluation module $L_z(\lambda^\natural)$ is real. Actually, it holds for all skew representations.

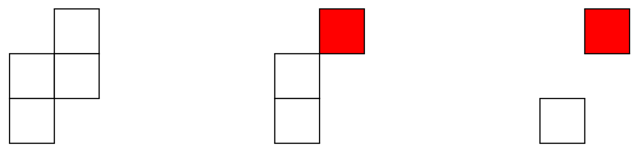
Theorem 4.20. *The skew representation $L_z(\lambda/\mu)$ is real.*

Proof. The statement follows from [NT02, Remark (d) of Theorem 4.8], [Naz04, Theorem 1.6], Theorem 4.3, and Proposition 4.8. □

4.6. **Application: extended T-systems.** In this section, we apply Drinfeld functor to show that the q-characters of skew representations satisfy extended T-systems.

Let $\mho = \lambda/\mu$ be a skew Young diagram. We say that $\mho$ is a *prime* skew Young diagram if it can not be divided into two parts intersecting at most one point.

Example 4.21. We explain the definition with the following 3 skew Young diagrams.



The first skew Young diagram is prime while the rest are not. For example, the last two diagrams can be divided into two parts so that one part is in red color. Clearly, the two parts of the second one intersects at a point while those of the third one are disconnected. □

Recall that different pairs (λ, μ) may give the same skew Young diagram. Let $\mho$ be a prime skew Young diagram (ignoring the content). We choose a λ so that λ and $\mho$ have the same number of columns and $\mho = \lambda/\mu$. The contents of $\mho$ are determined by λ, namely the box of λ at the left-upper corner has content zero. Let l be the number of columns of $\mho$, then $l = \lambda_1$.

Suppose $\mho = \lambda/\mu$ has at least two columns, namely $l \geqslant 2$. Let $\mho^+$ and $\mho^-$ be the prime skew Young diagrams obtained by deleting the leftmost column and the rightmost column of $\mho$, respectively. Also let $\mho^0$ be the prime skew Young diagrams obtained by removing both the leftmost and rightmost columns of $\mho$. Note that $\mho^0$ may be empty.

Define two skew Young diagrams $\mathbb{X}_\mho$ and $\mathbb{Y}_\mho$ as follows,

$$\mathbb{X}_\mho = \{(i,j) : \mu'_j + 1 \leqslant i \leqslant \lambda'_{j+1} - 1,\ 1 \leqslant j \leqslant l-1\},$$
$$\mathbb{Y}_\mho = \{(i,j) : \mu'_{j+1} \leqslant i \leqslant \lambda'_j,\ 1 \leqslant j \leqslant l-1\}.$$

The skew Young diagrams $\mathbb{X}_\mho$ and $\mathbb{Y}_\mho$ are obtained by taking the intersection and union, respectively, of the diagram $\mho^+$ shifted to the left by one unit and then up by one unit and $\mho^-$.

Note that in general as $\mho$ is prime, we have $\mu'_j \leqslant \lambda'_{j+1} - 1$ for $i = 1, \dots, l-1$. Hence the j-th column of $\mathbb{X}_\mho$ may be empty and $\mathbb{X}_\mho$ may be non-prime. However, $\mathbb{Y}_\mho$ is always prime.

Snakes defined in [MY12a] bijectively correspond to certain skew Young diagrams via the correspondence in [MY12a, Proposition 7.3]. The skew Young diagrams $\mathbb{X}_\mho$ and $\mathbb{Y}_\mho$ correspond to the neighbouring snakes in [MY12b, Section 3.6] in this sense.

Recall, if a skew Young diagram contains a rectangle of size $(m+1)\times(n+1)$ (a column of length $N+1$ in the $\mathrm{Y}(\mathfrak{gl}_N)$ case), then the corresponding skew representation has dimension zero.

Theorem 4.22 ([MY12b, Theorem 4.1])**.** *Suppose $\mho$ is a prime skew Young diagram having at least two columns and N is sufficiently large. Then we have the following relation in $\mathcal{R}ep(\mathrm{Y}(\mathfrak{gl}_N))$, the Grothendieck ring of the category of finite-dimensional representations of $\mathrm{Y}(\mathfrak{gl}_N)$,*

$$[L(\mho^+)] \otimes [L(\mho^-)] = [L(\mho^0)][L(\mho)] + [L(\mathbb{X}_\mho)][L(\mathbb{Y}_\mho)].$$

Moreover, $L(\mho^0) \otimes L(\mho)$ and $L(\mathbb{X}_\mho) \otimes L(\mathbb{Y}_\mho)$ are irreducible $\mathrm{Y}(\mathfrak{gl}_N)$-modules. □

Remark 4.23. Because we only care about the case when N is sufficiently large, our definitions of prime diagrams and $\mathbb{Y}_\mho$ here are slightly different from that in [MY12b, Section 3.5] as we allow a column of a prime skew Young diagram or $\mathbb{Y}_\mho$ to be very long. □

Applying Drinfeld functor, we get the corresponding supersymmetric version of Theorem 4.22.

Corollary 4.24. *Suppose $\mho$ is a prime skew Young diagram having at least two columns. Then we have the following relation in $\mathcal{R}ep(\mathcal{C})$,*

$$[L(\mho^+)]\otimes[L(\mho^-)]=[L(\mho^0)][L(\mho)]+[L(\mathbb{X}_\mho)][L(\mathbb{Y}_\mho)].$$

Moreover, $L(\mho^0)\otimes L(\mho)$ and $L(\mathbb{X}_\mho)\otimes L(\mathbb{Y}_\mho)$ are irreducible $\mathrm{Y}(\mathfrak{gl}_{m|n})$-modules.

Proof. Using Theorem 4.22 and Theorem 4.3 (the equivalence of Drinfeld functor), then we have the corresponding equality for the representations of degenerate affine Hecke algebra. Applying Drinfeld functor to this resulted equality, the first statement follows. Similarly, the second part follows from Proposition 4.8. □

Example 4.25. Given $i,j\in\mathbb{Z}_{>0}$ and $k\in\mathbb{C}$, let $\mho_{ij;k}$ be the rectangular Young diagram of size $i\times j$ whose left-upper corner box has content k. Then we have

$$\mho^+_{i(j+1);k}=\mho_{ij;k+1},\quad \mho^-_{i(j+1);k}=\mho_{ij;k},\quad \mho^0_{i(j+1);k}=\mho_{i(j-1);k+1},$$
$$\mathbb{X}_{\mho_{i(j+1);k}}=\mho_{(i-1)j;k},\qquad \mathbb{Y}_{\mho_{i(j+1);k}}=\mho_{(i+1)j;k+1}.$$

Let

$$\mathbf{T}^{(i)}_j(u+k-(i-j+1)/2)=\mathcal{K}_{\mho_{ij;k}}(u),$$

where $\mathcal{K}_{\mho_{ij}}(u)$ is the q-character of $L(\mho_{ij;k})$, see Theorem 3.4.

Setting $k=(i-j)/2$ and $\mho=\mho_{i(j+1);k}$, one obtains the T-systems from Corollary 4.24,

$$\mathbf{T}^{(i)}_j(u-\tfrac12)\mathbf{T}^{(i)}_j(u+\tfrac12)=\mathbf{T}^{(i)}_{j-1}(u)\mathbf{T}^{(i)}_{j+1}(u)+\mathbf{T}^{(i-1)}_j(u)\mathbf{T}^{(i+1)}_j(u).$$

The boundary conditions are given by

- $\mathbf{T}^{(i)}_j(u)=0$ if $i<0$ or $j<0$ or both $i>m$ and $j>n$;
- $\mathbf{T}^{(i)}_j(u)=1$ if $i,j\in\mathbb{Z}_{\geqslant0}$ and $ij=0$.

Hence we may regard Corollary 4.24 as extended T-systems. □

We remark that our extended T-systems are different from that of [Zha18, Theorem 3.3].

Suppose $\mho$ is a skew Young diagram with consecutive columns (not necessarily prime) and has at least 2 columns. One can define $\mho^\pm$ and $\mho^0$ in the same way.

Theorem 4.26. *If $\mho$ is not prime, then $L(\mho^+)\otimes L(\mho^-)$ and $L(\mho^0)\otimes L(\mho)$ are isomorphic and irreducible as $\mathrm{Y}(\mathfrak{gl}_{m|n})$-modules.*

Proof. The theorem is proved in a similar way to that of Corollary 4.24 using [MY12b, Theorem 4.3]. □

5. Quantum Berezinian and transfer matrices

5.1. **Quantum Berezinian.** Following [MR14], we recall the quantum Berezinian and related results in the case of $\mathrm{Y}(\mathfrak{gl}_{m|n})$.

Let $\mathcal{A}$ be a superalgebra. Let $a_{ij}\in\mathcal{A}$ with parity $|i|+|j|$. Suppose the inverse of the matrix A

$$A=\sum_{i,j\in\bar{I}}a_{ij}\otimes E_{ij}(-1)^{|i||j|+|j|}\in\mathcal{A}\otimes\mathrm{End}(V),$$

with values in $\mathcal{A}$ exists. Then we denote the entries of the inverse matrix by $a'_{ij}\in\mathcal{A}$:

$$A^{-1}=\sum_{i,j\in\bar{I}}a'_{ij}\otimes E_{ij}(-1)^{|i||j|+|j|}.$$

Define the *quantum Berezinian* $\mathrm{Ber}(A)$, see [Naz91], of the matrix A by

$$\mathrm{Ber}(A) = \sum_{\sigma\in\mathfrak{S}_m} \mathrm{sgn}(\sigma)\cdot a_{\sigma(1)1}\cdots a_{\sigma(m)m} \sum_{\tilde\sigma\in\mathfrak{S}_n} \mathrm{sgn}(\tilde\sigma)\cdot a'_{m+1,m+\tilde\sigma(1)}\cdots a'_{m+n,m+\tilde\sigma(n)}.$$

Let $\mathcal{A}_{m|n} := \mathrm{Y}(\mathfrak{gl}_{m|n})[[u^{-1}]]((\tau))$ be the superalgebra of Laurent series in τ whose coefficients are power series in u^{-1} whose coefficients are in $\mathrm{Y}(\mathfrak{gl}_{m|n})$ with the relations

$$(g_1u^{k_1}\tau^{l_1})(g_2u^{k_2}\tau^{l_2}) = g_1g_2u^{k_1}(u-l_1)^{k_2}\tau^{l_1+l_2}, \qquad g_1,g_2\in\mathrm{Y}(\mathfrak{gl}_{m|n}), l_1,l_2\in\mathbb{Z},\ k_1,k_2\in\mathbb{Z}_{\leqslant 0}.$$

Thus τ is the shift operator with respect to variable u and it should not be confused with automorphism of the Yangian τ_1 defined in (2.16).

Let q be a formal variable which commutes with all other elements. Let $A(q)$ be a matrix with elements in $\mathcal{A}_{m|n}[q]$ given by

$$A(q) = 1 - q\,T(u)\tau = \sum_{i,j\in\bar I}(\delta_{ij} - q\,t_{ij}(u)\tau)\otimes E_{ij}(-1)^{|i||j|+|j|}.$$

Clearly $A(q)$ is invertible. Let

$$\mathfrak{D}(u,\tau;q) := \mathrm{Ber}(A(q)) \tag{5.1}$$

be the quantum Berezinian. We simply write $\mathfrak{D}(u,\tau)$ for $\mathfrak{D}(u,\tau;1)$.

The matrix $T(u)$ is also invertible. Let $\mathfrak{Z}(u) = \mathrm{Ber}(T(u)\tau)\tau^{n-m}$. Note that $\mathfrak{Z}(u)$ does not contain τ. It is known the coefficients of $\mathfrak{Z}(u)$ generate the center of $\mathrm{Y}(\mathfrak{gl}_{m|n})$ and

$$\mathfrak{Z}(u) = \prod_{i\in\bar I}(d_i(u-\kappa_i))^{s_i},$$

in the notation of Proposition 3.5, see [Gow05, Theorem 1] and [Gow07, Theorem 4].

Recall the standard action of symmetric group $\mathfrak{S}_k$ on the space $V^{\otimes k}$ where σ_i acts as the graded flip operator $\mathcal{P}^{(i,i+1)}$, see (2.8). We denote by $\mathbb{A}_k$ and $\mathbb{S}_k$ the images of the normalized anti-symmetrizer and symmetrizer, respectively,

$$\mathbb{A}_k = \frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_k}\mathrm{sgn}(\sigma)\cdot\sigma, \qquad \mathbb{S}_k = \frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_k}\sigma.$$

Theorem 5.1 ([MR14, Theorem 2.13])**.** *We have*

$$\mathfrak{D}(u,\tau;q) = 1 + \sum_{k=1}^{\infty}(-1)^k \mathrm{str}\,\mathbb{A}_k\,T_1(u)T_2(u-1)\cdots T_k(u-k+1)q^k\tau^k, \tag{5.2}$$

$$\mathfrak{D}(u,\tau;q)^{-1} = 1 + \sum_{k=1}^{\infty}\mathrm{str}\,\mathbb{S}_k\,T_1(u)T_2(u-1)\cdots T_k(u-k+1)q^k\tau^k,$$

where the supertrace is taken over all copies of $\mathrm{End}(V)$. □

5.2. **Universal R-matrix and transfer matrices.** The Yangian $\mathrm{Y}(\mathfrak{gl}_{m|n})$ has a universal R-matrix. Its existence and properties can be deduced from [RS11], cf. also [Dri85]. We do not provide any justification in this paper.

The universal R-matrix is an element $\mathcal{R}(u) \in 1 + u^{-1}\mathrm{Y}(\mathfrak{gl}_{m|n})\otimes\mathrm{Y}(\mathfrak{gl}_{m|n})[[u^{-1}]]$ such that for all $X\in\mathrm{Y}(\mathfrak{gl}_{m|n})$ we have

$$(\mathrm{id}\otimes\Delta)(\mathcal{R}(u)) = \mathcal{R}_{12}(u)\mathcal{R}_{13}(u)\in\mathrm{Y}(\mathfrak{gl}_{m|n})^{\otimes 3}[[u^{-1}]],$$

$$(\Delta\otimes\mathrm{id})(\mathcal{R}(u)) = \mathcal{R}_{13}(u)\mathcal{R}_{23}(u)\in\mathrm{Y}(\mathfrak{gl}_{m|n})^{\otimes 3}[[u^{-1}]],$$

$$\mathcal{R}(u)\cdot(\mathrm{id}\otimes\tau_u)(\Delta^{\mathrm{op}}(X))=(\mathrm{id}\otimes\tau_u)(\Delta(X))\cdot\mathcal{R}(u)\in\mathrm{Y}(\mathfrak{gl}_{m|n})^{\otimes 2}[[u^{-1}]],$$

where τ_u is the Yangian automorphism defined in (2.16) (not to be confused with the shift operator τ). It follows that the universal R-matrix $\mathcal{R}(u)$ satisfies the Yang-Baxter equation

$$\mathcal{R}_{12}(u-v)\mathcal{R}_{13}(u)\mathcal{R}_{23}(v)=\mathcal{R}_{23}(v)\mathcal{R}_{13}(u)\mathcal{R}_{12}(u-v).$$

Let M be a finite-dimensional $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module. Denote by $\Theta_M:\mathrm{Y}(\mathfrak{gl}_{m|n})\to\mathrm{End}(M)$ the corresponding map. The R-matrix can be normalized so that

$$(\Theta_{\mathbb{V}_z}\otimes\mathrm{id})(\mathcal{R}(-u))=T(u+z)\in\mathrm{End}(V)\otimes\mathrm{Y}(\mathfrak{gl}_{m|n})[[u^{-1}]],$$

cf. [Naz98, Lemma 3.4 and Theorem 3.6].

Define the *transfer matrix* $\mathfrak{T}_M$ associated to M by

$$\mathfrak{T}_M(u)=\mathrm{str}_M\big((\Theta_M\otimes\mathrm{id})(\mathcal{R}(-u))\big)\in\mathrm{Y}(\mathfrak{gl}_{m|n})[[u^{-1}]].$$

The following lemma is standard, see [FR99, Lemma 2].

Lemma 5.2. *For any pair of finite-dimensional* $\mathrm{Y}(\mathfrak{gl}_{m|n})$*-modules* M_1 *and* M_2*, we have*

$$[\mathfrak{T}_{M_1}(u_1),\mathfrak{T}_{M_2}(u_2)]=0,\qquad \mathfrak{T}_{M_1\otimes M_2}(u)=\mathfrak{T}_{M_1}(u)\mathfrak{T}_{M_2}(u).$$

For a short exact sequence $M_1\hookrightarrow M\twoheadrightarrow M_2$*, we have* $\mathfrak{T}_M(u)=\mathfrak{T}_{M_1}(u)+\mathfrak{T}_{M_2}(u)$. □

Lemma 5.2 says that the map $\mathfrak{T}:\mathcal{R}ep(\mathcal{C})\to\mathrm{Y}(\mathfrak{gl}_{m|n})[[u^{-1}]]$ sending a finite-dimensional $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module M to the transfer matrix $\mathfrak{T}_M(u)$ in $\mathrm{Y}(\mathfrak{gl}_{m|n})[[u^{-1}]]$ is a ring homomorphism.

We shall focus on transfer matrices associated to skew representations $L(\lambda/\mu)^\iota$. When $M=L(\lambda/\mu)^\iota$, we write $\mathfrak{T}_{\lambda/\mu}(u)$ for $\mathfrak{T}_M(u)$. Then

$$\mathfrak{T}_{M_z}(u)=\mathfrak{T}_{\lambda/\mu}(u-z).$$

Recall that the partition (1^k) corresponds to the Young diagram consisting of a column with k boxes while (k) corresponds to the Young diagram consisting of a row with k boxes. We use the short-hand notation,

$$\mathfrak{T}^k(u):=\mathfrak{T}_{(1^k)}(u),\qquad \mathfrak{T}_k(u):=\mathfrak{T}_{(k)}(u).$$

Applying the map $\mathfrak{T}$ to the equality in $\mathcal{R}ep(\mathcal{C})$ corresponding to Theorem 3.16, we obtain the Jacobi-Trudi identity for transfer matrices.

Theorem 5.3. *Let* λ *and* μ *be two partitions such that* $\mu\subset\lambda$*. Then we have*

$$\begin{aligned}\mathfrak{T}_{\lambda/\mu}(u)&=\det_{1\leqslant i,j\leqslant\lambda_1'}\mathfrak{T}_{\lambda_i-\mu_j-i+j}(u+\mu_j-j+1)\\&=\det_{1\leqslant i,j\leqslant\lambda_1}\mathfrak{T}^{\lambda_i'-\mu_j'-i+j}(u-\mu_j'+j-1).\end{aligned}$$

Here we use the convention that $\mathfrak{T}_0(u)=\mathfrak{T}^0(u)=1$ *and* $\mathfrak{T}^k(u)=\mathfrak{T}_k(u)=0$ *for* $k<0$. □

Theorem 5.3 was conjectured in [Tsu97] on the level of eigenvalues, and proved for the case of hook Young diagrams in [KV08].

Corollary 5.4. *If* λ/μ *contains a rectangle of size at least* $(m+1)\times(n+1)$*, then*

$$\det_{1\leqslant i,j\leqslant\lambda_1'}\mathfrak{T}_{\lambda_i-\mu_j-i+j}(u+\mu_j-j+1)=\det_{1\leqslant i,j\leqslant\lambda_1}\mathfrak{T}^{\lambda_i'-\mu_j'-i+j}(u-\mu_j'+j-1)=0.$$

Proof. The statement follows from Corollary 3.18. □

Proposition 5.5. *We have*

$$\mathfrak{T}^k(u) = \operatorname{str} \mathbb{A}_k\, T_1(u)T_2(u-1)\cdots T_k(u-k+1),$$

$$\mathfrak{T}_k(u) = \operatorname{str} \mathbb{S}_k\, T_1(u)T_2(u+1)\cdots T_k(u+k-1).$$

Proof. Consider the weight ω_k. Then the corresponding Young diagram is a column with k boxes. Let Ω be the column tableau. Then $c_i(\Omega) = 1-i$, $i = 1,\dots,k$, and it is well-known that $\mathcal{E}_\Omega = k!\mathbb{A}_k$ and $\mathbb{A}_k$ is the projection of $V^{\otimes k}$ onto the image of $\mathcal{E}_\Omega$, which is isomorphic to $L(\omega_k)$ as a $\mathfrak{gl}_{m|n}$-module. The first formula of the proposition now follows from Theorem 4.14. The second formula is similar. □

5.3. **Harish-Chandra homomorphism.** In this section, we define an analog of Harish-Chandra homomorphism $\mathcal{H}$ for $\mathrm{Y}(\mathfrak{gl}_{m|n})$ and compute the images of transfer matrices associated to skew Young diagrams under $\mathcal{H}$.

Let $\mathrm{Y}(\mathfrak{gl}_{m|n})^{\mathfrak{h}}$ be the centralizer of $\mathfrak{h} \subset \mathfrak{gl}_{m|n}$ in $\mathrm{Y}(\mathfrak{gl}_{m|n})$,

$$\mathrm{Y}(\mathfrak{gl}_{m|n})^{\mathfrak{h}} = \{X \in \mathrm{Y}(\mathfrak{gl}_{m|n}) \mid [t_{ii}^{(1)}, X] = 0, \ \text{ for } i \in \bar{I}\}.$$

Let J be the intersection of $\mathrm{Y}(\mathfrak{gl}_{m|n})^{\mathfrak{h}}$ and the right ideal of $\mathrm{Y}(\mathfrak{gl}_{m|n})$ generated by $f_j^{(r)}$, for $j \in I$ and $r \in \mathbb{Z}_{>0}$. Note that J is also the intersection of $\mathrm{Y}(\mathfrak{gl}_{m|n})^{\mathfrak{h}}$ and the left ideal of $\mathrm{Y}(\mathfrak{gl}_{m|n})$ generated by $e_j^{(r)}$, for $j \in I$ and $r \in \mathbb{Z}_{>0}$. We have the decomposition as vector spaces,

$$\mathrm{Y}(\mathfrak{gl}_{m|n})^{\mathfrak{h}} = \mathrm{Y}^0_{m|n} \oplus J.$$

The projection $\mathcal{H}$ of $\mathrm{Y}(\mathfrak{gl}_{m|n})^{\mathfrak{h}}$ onto the subspace $\mathrm{Y}^0_{m|n}$ along J,

$$\mathcal{H} : \mathrm{Y}(\mathfrak{gl}_{m|n})^{\mathfrak{h}} \to \mathrm{Y}^0_{m|n},$$

is an algebra homomorphism. We call $\mathcal{H}$ the *Harish-Chandra homomorphism* of $\mathrm{Y}(\mathfrak{gl}_{m|n})$.

Clearly, from the Gauss decomposition, we have $\mathcal{H}(t_{ii}(u)) = d_i(u)$.

The following lemma can be proved using induction, cf. [MTV06, Lemma 4.1].

Lemma 5.6. *For any $X \in \mathrm{Y}(\mathfrak{gl}_{m|n})^{\mathfrak{h}}$ of the form*

$$* \, t_{i_0j_0}^{(r_0)} t_{i_1i_1}^{(r_1)} \cdots t_{i_ki_k}^{(r_k)}, \quad i_a, j_0 \in \bar{I},\ r_a > 0,\ a = 0,1,\dots,k,\ k \in \mathbb{Z}_{\geqslant 0}, \quad i_0 < j_0,$$

where $*$ is an element in $\mathrm{Y}(\mathfrak{gl}_{m|n})$, we have $\mathcal{H}(X) = 0$. □

It is well-known that all coefficients of $\mathfrak{T}^k(u)$ and $\mathfrak{T}_k(u)$ belong to the centralizer $\mathrm{Y}(\mathfrak{gl}_{m|n})^{\mathfrak{h}}$, cf. [MTV06, Proposition 4.7]. Hence we can compute the Harish-Chandra images of transfer matrices $\mathfrak{T}_{\lambda/\mu}(u)$.

Lemma 5.7. *We have*

$$\mathcal{H}(\mathfrak{T}^k(u)) = \sum_{\mathcal{I}} \prod_{a=1}^{k} s_{i_a} d_{i_a}(u-a+1),$$

$$\mathcal{H}(\mathfrak{T}_k(u)) = \sum_{\mathcal{J}} \prod_{a=1}^{k} s_{j_a} d_{j_a}(u-a+k),$$

summed over all sequences $\mathcal{I} = \{1 \leqslant i_1 < i_2 < \cdots < i_b < m+1 \leqslant i_{b+1} \leqslant \cdots \leqslant i_k \leqslant m+n\}$ and $\mathcal{J} = \{m+n \geqslant j_1 > j_2 > \cdots > j_b \geqslant m+1 > j_{b+1} \geqslant \cdots \geqslant j_k \geqslant 1\}$ with $b = 0,\dots,k$, respectively.

Proof. The lemma follows from Proposition 5.5, Lemma 5.6, and an analog of [MR14, Proposition 2.3] with sequences of the form like $\mathcal{I}$ or $\mathcal{J}$, see [MR14, Remark 2.4]. A similar computation was done in [MR14, Section 3.3]. □

The following is immediate from Theorem 5.1 and Lemma 5.7.

Corollary 5.8. *The Harish-Chandra image of* $\mathfrak{D}(u,\tau;q)$ *is given by*

$$\mathcal{H}(\mathfrak{D}(u,\tau;q)) = \prod_{1\leqslant i\leqslant m+n}^{\longrightarrow} \Big(1 - q\, d_i(u)\,\tau\Big)^{s_i}. \qquad \square$$

Proposition 5.9. *We have*

$$\mathcal{H}(\mathfrak{T}_{\lambda/\mu}(u)) = \sum_{\mathcal{T}} \prod_{(i,j)\in\lambda/\mu} s_{\mathcal{T}(i,j)} d_{\mathcal{T}(i,j)}(u + c(i,j)), \tag{5.3}$$

where the summation is over all semi-standard Young tableaux $\mathcal{T}$ *of shape* λ/μ.

Proof. The statement follows from the fact that $\mathcal{H}$ is an algebra homomorphism, Theorem 5.3, Lemma 5.7 and the proof of Theorem 3.16 (identifying $s_i d_i(u+a)$ with $\mathcal{X}_{i,a}$). $\square$

Remark 5.10. Note that if we identify $s_i d_i(u+a)$ with $\mathcal{X}_{i,a}$ for $i\in\bar{I}$ and $a\in\mathbb{C}$, where s_i is the parity of $\mathcal{X}_{i,a}$, then the right hand side of (5.3) is identified with right hand side of the equation in Theorem 3.4. This implies that the q-character map can be thought as the composition of Harish-Chandra map and the map $\mathfrak{T}$, see [FR99, Section 3]. $\square$

Corollary 5.11. *If* λ/μ *does not contain a rectangle of size* $(m+1)\times(n+1)$*, then* $\mathcal{H}(\mathfrak{T}_{\lambda/\mu}(u))$ *is non-zero. In particular,* $\mathfrak{T}_{\lambda/\mu}(u)$ *is non-zero.*

Proof. As λ/μ does not contain a rectangle of size $(m+1)\times(n+1)$, there exists at least one semi-standard Young tableau of shape λ/μ. Hence the space $L(\lambda/\mu)$ is non-trivial by Theorem 3.4 and an irreducible finite-dimensional $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module by Theorem 4.9. Let $\mathcal{T}_0$ be the semi-standard Young tableau corresponding to the highest ℓ-weight of $L(\lambda/\mu)$, see Theorem 3.4. We consider the monomial

$$\prod_{(i,j)\in\lambda/\mu} \Big(d^{(1)}_{\mathcal{T}_0(i,j)} u^{-1}\Big)$$

in the right hand side of (5.3). Since $\mathcal{T}_0$ also corresponds to the highest $\mathfrak{gl}_{m|n}$-weight in $L(\lambda/\mu)$, this monomial appears only in

$$\prod_{(i,j)\in\lambda/\mu} s_{\mathcal{T}(i,j)} d_{\mathcal{T}(i,j)}(u + c(i,j))$$

when $\mathcal{T}=\mathcal{T}_0$. The statement now follows from the fact that $d_i^{(r)}$, $i\in\bar{I}$ and $r\in\mathbb{Z}_{>0}$, are algebraically independent. $\square$

5.4. **Rational form of quantum Berezinian.** Motivated by [HMVY19, HLM19] and [LM19, Corollary 6.13], we are interested in writing $\mathrm{Ber}(1-T(u)\tau)$ as a ratio of two polynomials in τ.

Let Ξ, Ξ^+, and Ξ^- be partitions corresponding to the rectangular Young diagrams of sizes $m\times n$, $m\times(n+1)$, and $(m+1)\times n$, respectively. Introduce partitions

$$\Upsilon_i^+ = (\underbrace{n,\dots,n}_{m\ n'\mathrm{s}}, i), \qquad \Upsilon_j^- = (\underbrace{n+1,\dots,n+1}_{j\ (n+1)'\mathrm{s}}, \underbrace{n,\dots,n}_{(m-j)\ n'\mathrm{s}}),$$

where $i=0,1,\dots,n$ and $j=1,\dots,m$. In particular, $\Upsilon_0^+=\Upsilon_0^-=\Xi$, $\Upsilon_n^+=\Xi^-$, $\Upsilon_m^-=\Xi^+$.

Note that $\mathfrak{T}_\Xi(u)\neq 0$ by Corollary 5.11.

Theorem 5.12. *We have*

$$\mathfrak{D}(u,\tau;q)=\Big(1+\sum_{i=1}^{m}(-1)^i q^i \mathcal{E}_i(u)\tau^i\Big)\Big(1+\sum_{j=1}^{n} q^j \mathcal{G}_j(u)\tau^j\Big)^{-1},$$
$$=\Big(1+\sum_{j=1}^{n} q^j \overline{\mathcal{G}}_j(u)\tau^j\Big)^{-1}\Big(1+\sum_{i=1}^{m}(-1)^i q^i \overline{\mathcal{E}}_i(u)\tau^i\Big).$$

where

$$\mathcal{E}_i(u)=\frac{\mathfrak{T}_{\Xi^+/(1^{m-i})}(u+m-i)}{\mathfrak{T}_{\Xi}(u+m+1-i)},\qquad \mathcal{G}_i(u)=\frac{\mathfrak{T}_{\Upsilon_i^+}(u+m+1-i)}{\mathfrak{T}_{\Xi}(u+m+1-i)},$$
$$\overline{\mathcal{E}}_i(u)=\frac{\mathfrak{T}_{\Upsilon_i^-}(u-n)}{\mathfrak{T}_{\Xi}(u-n)},\qquad \overline{\mathcal{G}}_i(u)=\frac{\mathfrak{T}_{\Xi^-/(m-i)}(u-n+1)}{\mathfrak{T}_{\Xi}(u-n)},$$

are ratios of transfer matrices. Here we can take ratio as transfer matrices commute.

Proof. We only show the first equality for $q=1$. The second one and the case of general q are similar.

By Theorem 5.1 and Proposition 5.5, it suffices to show that

$$\Big(1+\sum_{k=1}^{\infty}(-1)^k\mathfrak{T}^k(u)\tau^k\Big)\Big(1+\sum_{j=1}^{n}\mathcal{G}_j(u)\tau^j\Big)=1+\sum_{i=1}^{m}(-1)^i\mathcal{E}_i(u)\tau^i.$$

This reduces to show that

$$\mathcal{E}_i(u)=\mathfrak{T}^i(u)+\sum_{a=1}^{\min(i,n)}(-1)^a\mathfrak{T}^{i-a}(u)\mathcal{G}_a(u-i+a),\qquad i=1,\dots,m; \tag{5.4}$$

$$0=\mathfrak{T}^j(u)+\sum_{a=1}^{\min(j,n)}(-1)^a\mathfrak{T}^{j-a}(u)\mathcal{G}_a(u-j+a),\qquad j=m+1,m+2,\dots. \tag{5.5}$$

Let us first show equation (5.4). By Theorem 5.3, we have

$$\mathfrak{T}_{\Xi^+/(1^{m-i})}(u+m-i)=\begin{vmatrix}\mathfrak{T}^i(u) & \mathfrak{T}^{m+1}(u+m-i+1) & \cdots & \mathfrak{T}^{m+n}(u+m-i+n)\\ \mathfrak{T}^{i-1}(u) & \mathfrak{T}^{m}(u+m-i+1) & \cdots & \mathfrak{T}^{m+n-1}(u+m-i+n)\\ \vdots & \vdots & \ddots & \vdots\\ \mathfrak{T}^{i-n}(u) & \mathfrak{T}^{m+1-n}(u+m-i+1) & \cdots & \mathfrak{T}^{m}(u+m-i+n)\end{vmatrix} \tag{5.6}$$

and

$$\mathfrak{T}_{\Xi}(u+m+1-i)=\begin{vmatrix}\mathfrak{T}^{m}(u+m-i+1) & \cdots & \mathfrak{T}^{m+n-1}(u+m-i+n)\\ \vdots & \ddots & \vdots\\ \mathfrak{T}^{m+1-n}(u+m-i+1) & \cdots & \mathfrak{T}^{m}(u+m-i+n)\end{vmatrix}. \tag{5.7}$$

It follows from Theorem 5.3 that $\mathfrak{T}_{\Upsilon_a^+}(u+m+1-i)$ is equal to the minor of the matrix in (5.6) obtained by deleting the first column and the $(a+1)$-th row. Expanding the determinant in (5.6) with respect to the first column and dividing both sides by the determinant in (5.7), then equation (5.4) follows from (5.7).

Denote by $\mathfrak{X}_i(u)$ the determinant in (5.6).

Equation (5.5) is proved similarly for $i=m+1,\dots,m+n$. In this case we have $\mathfrak{X}_i(u)=0$ as the first and $(i+1-m)$-th columns coincide. Hence equation (5.5) is obtained by using Theorem 5.3 and expanding $\mathfrak{X}_i(u)=0$ with respect to the first column.

Finally, we show equation (5.5) for $i=m+n+k$ where $k\geqslant 1$. Let λ and μ be partitions corresponding to rectangular Young diagrams of sizes $(m+k)\times(n+1)$ and $(k-1)\times n$, respectively. Clearly, λ/μ

contains a rectangle of size $(m+1)\times(n+1)$. It follows from Corollary 5.4 that $\mathfrak{T}_{\lambda/\mu}(u)=0$. Applying Theorem 5.3, we have

$$0=(-1)^n\mathfrak{T}_{\lambda/\mu}(u-n)=\mathfrak{X}_i(u).$$

Again, equation (5.5) is obtained by expanding the determinant $\mathfrak{X}_i(u)$ with respect to the first column. □

Recall that $\mathbb{C}^{(p)}_{1+\frac{1}{u}}$ is the one-dimensional $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module generated by a vector of highest ℓ-weight $(1+\frac{1}{u},\dots,1+\frac{1}{u})$ of parity p.

Corollary 5.13. *We have*

$$\mathfrak{Z}(u)=(-1)^n\frac{\mathfrak{T}_{\Xi^+}(u)}{\mathfrak{T}_{\Xi^-}(u+1)}=\mathfrak{T}_{\mathbb{C}^{(\bar 0)}_{1+\frac{1}{u}}}.$$

Proof. Note that $(-q)^{n-m}\mathrm{Ber}(1-q\,T(u)\tau)\to\mathfrak{Z}(u)\tau^{m-n}$ as $q\to\infty$. The first equality follows from Theorem 5.12 by taking the limit $q\to\infty$. It is easy to see that

$$L_0\big((\Xi^+)^\natural\big)\cong L_{-1}\big((\Xi^-)^\natural\big)\otimes\mathbb{C}^{(\bar n)}_{1+\frac{1}{u}}.$$

The second equality follows since $\mathfrak{T}$ is a homomorphism of rings. □

5.5. **Spectra of transfer matrices and divisibility of q-characters.** In this section, we describe the relation between Theorem 5.12 and the results in [HLM19].

Let M be a finite-dimensional irreducible $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module of highest ℓ-weight $\boldsymbol{\zeta}=(\zeta_i(u);s)_{i\in\bar I}$. We are interested in finding the spectra of transfer matrices acting on the space M.

Let $\boldsymbol{l}=(l_i)_{i\in I}$ be a sequence of non-negative integers. Let $\boldsymbol{t}=(t_j^{(i)})$, $i\in I$, $j=1,\dots,l_i$, be a sequence of complex numbers. Define monic polynomials

$$y_i(u)=\prod_{j=1}^{l_i}(u-t_j^{(i)}),$$

and set $\boldsymbol{y}=(y_i)_{i\in I}$.

The *Bethe ansatz equation* associated to $\boldsymbol{\zeta}$, $\boldsymbol{l}$, $(s_i)_{i\in\bar I}$ is a system of algebraic equations in $\boldsymbol{t}$ given by

$$\frac{\zeta_i(t_j^{(i)})}{\zeta_{i+1}(t_j^{(i)})}\frac{y_{i-1}(t_j^{(i)}+s_i)}{y_{i-1}(t_j^{(i)})}\frac{y_i(t_j^{(i)}-s_i)}{y_i(t_j^{(i)}+s_{i+1})}\frac{y_{i+1}(t_j^{(i)})}{y_{i+1}(t_j^{(i)}-s_{i+1})}=1, \tag{5.8}$$

where $i\in I$ and $j=1,\dots,l_i$. It is known that when the Bethe ansatz equation is satisfied, one can construct the *Bethe vector* $\mathbb{B}_{\boldsymbol{l}}(\boldsymbol{t})\in M$ which is shown to be an eigenvector (if it is nonzero) of the first transfer matrix $\mathrm{str}(T(u))$, see [BR08]. One also expects the Bethe vector to be an eigenvector of all transfer matrices. Motivated by [HLM19], we have the following conjecture.

Let $y_0(u)=y_{m+n}(u)=1$. Define a rational difference operator $\mathfrak{D}(u,\tau,\boldsymbol{\zeta},\boldsymbol{y};q)$ by

$$\mathfrak{D}(u,\tau,\boldsymbol{\zeta},\boldsymbol{y};q)=\prod_{1\leqslant i\leqslant m+n}^{\longrightarrow}\Big(1-q\,\zeta_i(u)\cdot\frac{y_{i-1}(u+s_i)y_i(u-s_i)}{y_{i-1}(u)y_i(u)}\,\tau\Big)^{s_i}. \tag{5.9}$$

Conjecture 5.14. *If $\boldsymbol{t}$ satisfies the Bethe ansatz equation (5.8), then we have*

$$\mathfrak{D}(u,\tau;q)\,\mathbb{B}_{\boldsymbol{l}}(\boldsymbol{t})=\mathfrak{D}(u,\tau,\boldsymbol{\zeta},\boldsymbol{y};q)\,\mathbb{B}_{\boldsymbol{l}}(\boldsymbol{t}).$$ □

The conjecture was confirmed for $\mathrm{Y}(\mathfrak{gl}_N)$ in [MTV06, Theorem 6.1] and for $\mathrm{Y}(\mathfrak{gl}_{1|1})$ in [LM19, Theorem 6.5]. Conjecture 5.14 can be thought as the supersymmetric version of [FH15, Theorem 5.11] and [FJMM17, Theorem 7.5]. Namely, the eigenvalues of transfer matrix associated to a finite dimensional $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module W acting on the finite dimensional $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module M can be obtained by certain substitutions to the Harish-Chandra image of $\mathfrak{T}_W(u)$. For instance, applying the substitutions

$$d_i(u) \mapsto \zeta_i(u) \cdot \frac{y_{i-1}(u+s_i)y_i(u-s_i)}{y_{i-1}(u)y_i(u)} \tag{5.10}$$

to the Harish-Chandra image $\mathcal{H}(\mathfrak{D}(u,\tau;q))$ in Corollary 5.8, one obtains exactly the rational difference operator $\mathfrak{D}(u,\tau,\boldsymbol{\zeta},\boldsymbol{y};q)$ in (5.9).

The rational difference operator on the right hand side of (5.9) can also be understood using Theorem 5.12 and the divisibility of q-characters in Section 3.3. Let

$$\mathfrak{D}_1(u,\tau,\boldsymbol{\zeta},\boldsymbol{y};q) = \prod_{1\leqslant i\leqslant m}^{\longrightarrow} \Big(1 - q\,\zeta_i(u)\cdot \frac{y_{i-1}(u+s_i)y_i(u-s_i)}{y_{i-1}(u)y_i(u)}\,\tau\Big),$$

$$\mathfrak{D}_2(u,\tau,\boldsymbol{\zeta},\boldsymbol{y};q) = \prod_{m+1\leqslant i\leqslant m+n}^{\longleftarrow} \Big(1 - q\,\zeta_i(u)\cdot \frac{y_{i-1}(u+s_i)y_i(u-s_i)}{y_{i-1}(u)y_i(u)}\,\tau\Big).$$

Then

$$\mathfrak{D}(u,\tau,\boldsymbol{\zeta},\boldsymbol{y};q) = \mathfrak{D}_1(u,\tau,\boldsymbol{\zeta},\boldsymbol{y};q)\big(\mathfrak{D}_2(u,\tau,\boldsymbol{\zeta},\boldsymbol{y};q)\big)^{-1}. \tag{5.11}$$

The rational form decomposition of $\mathfrak{D}(u,\tau;q)$ in Theorem 5.12 is consistent with that of $\mathfrak{D}(u,\tau,\boldsymbol{\zeta},\boldsymbol{y};q)$ in (5.11) in the following sense. We compute the Harish-Chandra images of $\mathcal{E}_i(u)$ in Theorem 5.12. Consider $\mathcal{H}(\mathfrak{T}_{\lambda/\mu}(u))$ as a polynomial in formal variables $d_i(u+a)$ for $i\in\bar{I}$ and $a\in\mathbb{C}$, see Proposition 5.9. It follows from Lemma 3.12 and Proposition 5.9 that polynomial $\mathcal{H}(\mathfrak{T}_{\Xi}(u+m+1-i))$ divides polynomial $\mathcal{H}(\mathfrak{T}_{\Xi^+/(1^{m-i})}(u+m-i))$ and the quotient is

$$\mathcal{H}(\mathcal{E}_i(u)) = \sum_{1\leqslant j_1<\cdots<j_i\leqslant m}\prod_{a=1}^{i} d_{j_a}(u-a+1).$$

Similarly, we have

$$\mathcal{H}(\mathcal{G}_i(u)) = \sum_{1\leqslant j_1<\cdots<j_i\leqslant n}(-1)^i\prod_{a=1}^{i} d_{m+j_a}(u-i+a).$$

Applying substitutions (5.10) again, we have

$$\mathcal{H}\Big(1+\sum_{i=1}^{m}(-1)^i q^i\mathcal{E}_i(u)\tau^i\Big)\mapsto \mathfrak{D}_1(u,\tau,\boldsymbol{\zeta},\boldsymbol{y};q),\quad \mathcal{H}\Big(1+\sum_{j=1}^{n}q^j\mathcal{G}_j(u)\tau^j\Big)\mapsto \mathfrak{D}_2(u,\tau,\boldsymbol{\zeta},\boldsymbol{y};q).$$

Remark 5.15. There is a similar explanation for the second rational form decomposition of $\mathfrak{D}(u,\tau;q)$ in Theorem 5.12 using the divisibility of q-characters in Section 3.3. However, in this case, it is known from [HMVY19, HLM19] that the corresponding parity sequence is $(-1,\dots,-1,1,\dots,1)$ (the number -1 occurs n times while 1 occurs m times) instead of the standard parity sequence $(1,\dots,1,-1,\dots,-1)$. For the semi-standard Young tableaux, we fill in numbers $1,\dots,n$ strictly increasing along rows and weakly increasing along columns while $n+1,\dots,n+m$ are filled in strictly increasing along columns and weakly increasing along rows. Then it is easy to show similar divisibility equalities of q-characters (Lemma 3.12) for modules corresponding to the skew Young diagrams involved in $\overline{\mathcal{E}}_i(u)$ and $\overline{\mathcal{G}}_i(u)$ from Theorem 5.12. □

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