Physics notes for Math 120

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Math 120 does a lot with vector fields, line and surface integrals, and theorems that describe relationships between these. These come up naturally in physics, and will be included in Math 120 as the natural examples of the mathematics we study.

Written up here are some supplemental notes on the underlying physics, examples of vector fields with pictures, and Maxwell's equations in differential and integral form. You will not be responsible for any of the physics on Math 120 exams: these notes are meant to provide some background for those of you who are interested.

Enjoy :)

Miki

PS: Let me know if you find any errors (bonus points!) (well, maybe chocolate at least)

Section 16.1: Electric and magnetic fields

Electric fields

Coulomb's law gives the force between two charges:

$$\vec{F} = \frac{q_1 q_2}{4\pi\epsilon_0 r^2}\hat{r}$$

Here $\epsilon_0 \approx 8.85 \times 10^{-12} F/m$ (Farads per meter) is the electric constant, or permittivity of free space. Direction of the force depends on signs of the charges: opposite charges attract, same charges repel.

An electric field $\vec{E}$ acts on a particle with charge q. The field pushes the particle along the field lines, in one or the other direction, depending on the charge:

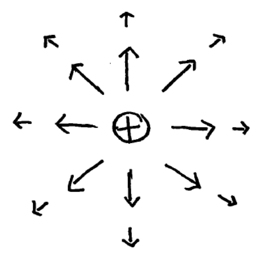
$$\vec{F} = q\vec{E}$$

We can measure electric fields this way: put a charge in, find the force, divide by charge.
Here are the most basic examples of electric fields.

Example 1: Single positive charge

Put a positive charge q at the origin. The electric field $\vec{E}$ points outwards, with strength that falls off with square of distance. Formula:

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$$

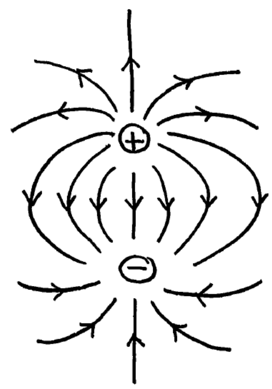


Example 2: Dipole

A dipole is made up of charges q and $-q$, with vector $\vec{s}$ being the displacement vector from negative to positive charge. Define $\vec{p} = q\vec{s}$. The electric field is

$$E = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}[3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}]$$

This is not a particularly nice formula, but the picture is quite famous and worth seeing. It will come up again for magnetic fields.



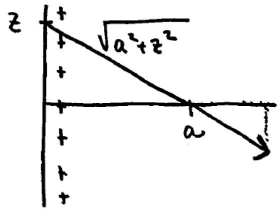
For a continuous charge distribution dQ (sheet, wire, etc.), the formula becomes

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\int \frac{dQ}{|\vec{R}|^2}\hat{R}$$

where $\vec{R}$ points from a given spot on the charged object to your position. Usually we deal with symmetric cases where the answer is reasonably simple, as in the examples below.

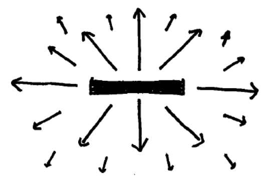
Example 3: Charged wire

Let's take a wire with a constant positive charge distribution λ. We shall position it along z-axis, from $-L$ to L. We'll stand on the x-axis at distance a from the wire. The vertical components of the field cancel out, and what's left is a horizontal field pointing radially out. To get the strength of the field, use the charge distribution formula with $dQ = \lambda dz$. Note that the component of $\hat{R}$ in the x-direction contributed from z is $a/\sqrt{a^2+z^2}$.

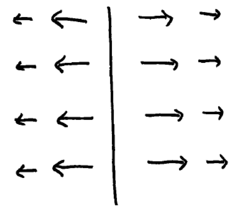


$$E = \frac{1}{4\pi\epsilon_0}\int_{-L}^{L}\frac{a\lambda}{(a^2+z^2)^{3/2}}dz = \frac{a\lambda}{4\pi\epsilon_0}\left.\frac{z}{a^2\sqrt{a^2+z^2}}\right|_{-L}^{L} = \frac{\lambda L}{2\pi\epsilon_0 a\sqrt{a^2+L^2}}$$

Away from the xy-plane, the field will not be horizontal, since the situation is no longer symmetric. On the right is a picture of the resulting field. The calculation of both components is very similar to the x-axis case, only longer and less pretty.



We shall restrict ourselves to the case of an infinite wire, which is symmetric from every point, as there is an infinite stretch of wire in both directions. The field is therefore horizontal, and its strength can be gotten by taking the limit $L \to \infty$ in the calculation we have done above:

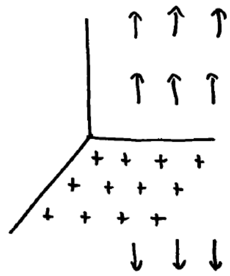


$$E = \frac{\lambda}{2\pi\epsilon_0 a}$$

Note that this only decreases as $1/a$ with distance a from the wire, as opposed to $1/a^2$ for a single charge: the contributions from the infinite wire add up.

Example 4: Charged sheet

We take an infinite sheet lying on the xy-plane, with a constant positive charge distribution σ. We sit above the sheet, on the z-axis, at distance a. This time all the horizontal components cancel out, and the resulting field is vertical. To find it, we can use the charge distribution formula with $dQ = \sigma dA = \sigma r dr d\theta$. The vertical component of $\hat{R}$ for the integral is $a/\sqrt{a^2+r^2}$, in polar coordinates.



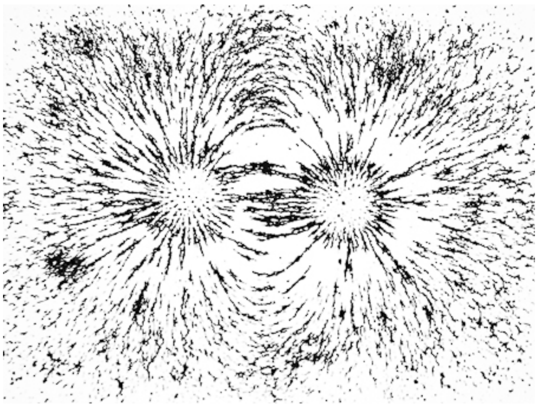
$$E = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{\infty}\frac{a\sigma r}{(a^2+r^2)^{3/2}}dr d\theta = \frac{a\sigma}{4\pi\epsilon_0}2\pi\left.\left[-\frac{1}{\sqrt{d^2+r^2}}\right]\right|_0^{\infty} = \frac{\sigma}{2\epsilon_0}$$

This formula is independent on the distance from the sheet: the field has the same strength everywhere, no more decreasing with distance at all. The contributions from an infinite two dimensional sheet of charges add up to make the field constant.

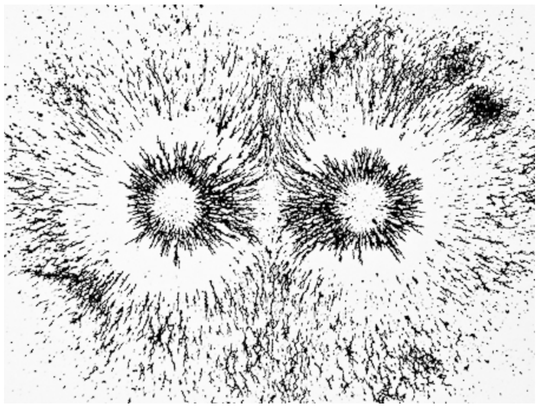
Magnetic fields

A magnetic field picture for a straight magnet looks just like a dipole, with N and S in place of q and $-q$, respectively. But: there is no such thing as a magnetic charge - at least not that we have ever found. Magnets do have two poles, but cutting a magnet in half just gives two magnets. It's full of tiny electron magnets, that create a net magnetic field when they are aligned.

Here are two pictures of magnetic field lines: we created them by pouring iron filings on top of two magnets. In the left picture, one magnet has N facing up and the other one S, so they attract, and the field lines go from one to the other. In the second picture, both magnets have N facing up, so they repel. See if you can figure out the configurations on the next page.



Magnetic field lines: NS.

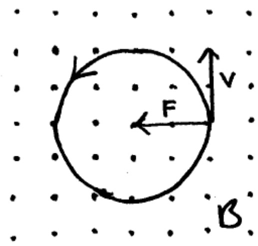


Magnetic field lines: NN.

What happens if a particle of charge q is sitting in a magnetic field? Nothing. To get a force, you need the particle to be moving. As if that were not strange enough, the force on this particle will be perpendicular to its velocity $\vec{v}$, and to $\vec{B}$ itself:

$$\vec{F} = q\vec{v} \times \vec{B}$$

For example, let us look at a constant magnetic field $\vec{B}$, pointing into the paper: the field lines are symbolized by dots in the picture: think of them as lines piercing the page. A particle traveling up in the page will have force $\vec{F}$ on it pointing left. This has no effect on the particle's speed, but changes the direction of the velocity to the left. The force changes accordingly, always perpendicular to the path, but of constant magnitude, and the particle ends up traveling on a circular path.



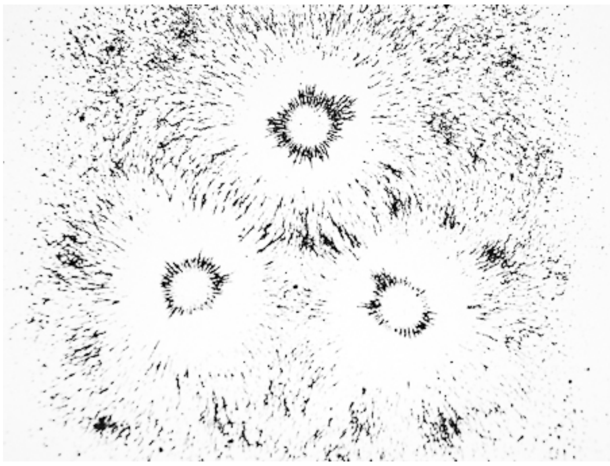
In the presence of both electric and magnetic fields, the combined force on a charged particle is given by the Lorentz force law:

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B} = q(\vec{E} + \vec{v} \times \vec{B})$$

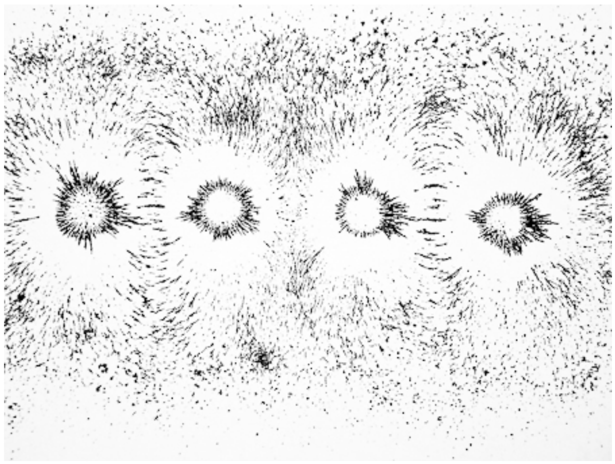
This is a way to get a new vector field from old ones. Thomson took these and tuned them to make the net force to be zero: this was part of his cathode ray tube experiments, which led him to the discovery of the electron. Note that to get zero, $\vec{E}$ has to be perpendicular to $\vec{B}$, they are referred to as "crossed fields".

Magnetic game

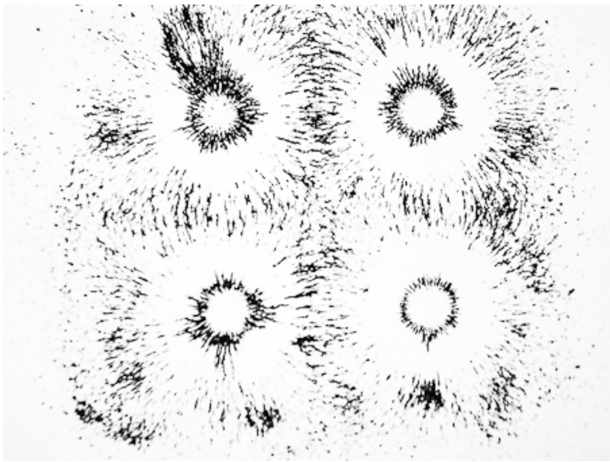
Try a little game with magnetic lines. Each picture has one of the magnets labeled. Can you tell what the other labels are? Answer is in footnote[1]. Don't read it right away!



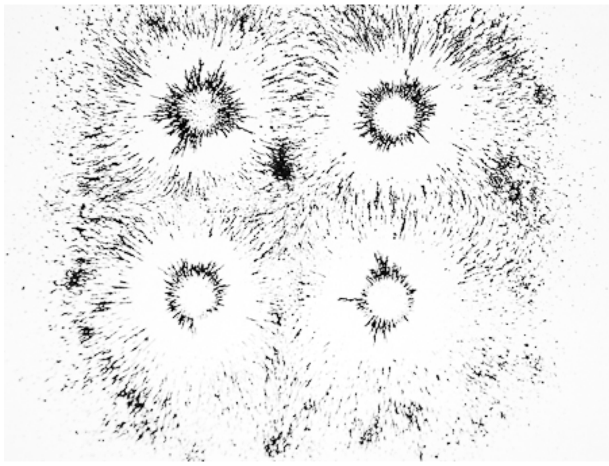
Picture 1. Top magnet: N.



Picture 2. Left magnet: N.



Picture 3. Top left magnet: N.



Picture 4. Top left magnet: N.

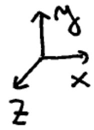
[1] Reading row by row, left to right. Pic 1: NSS. Pic2: NSSN. Pic3: NSSN. Pic4: NSNN.

16.5: Differential form of Maxwell's equations

For this section, it is useful to keep in mind two main examples of fields (as far as div and curl are concerned).

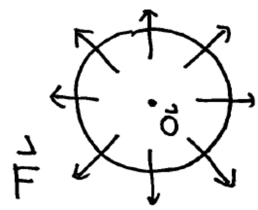
The first one is an outward field

$$\vec{F} = (x, y, z)$$



This is really *the* div field, with positive div and zero curl:

$$\nabla \cdot \vec{F} = 1 + 1 + 1 = 3, \quad \nabla \times \vec{F} = (0, 0, 0)$$

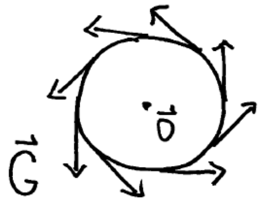


The second example is a field that rotates counterclockwise around the z-axis:

$$\vec{G} = (-y, x, 0)$$

This one is *the* curl field, with zero div and curl pointing straight up.

$$\nabla \cdot \vec{G} = 0 + 0 + 0 = 0, \quad \nabla \times \vec{G} = (0, 0, 2)$$



This is where div and curl got their names: Div measures the outward spread of the field, ignoring the rotation. Curl measures rotation, ignoring the outward component. Curl also gives the axis of rotation: in case of our $\vec{G}$, it points along the z-axis.

Gauss' law for electric fields:

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

This basically says that the div of E at a point is proportional to charge density ρ. We have seen that the electric field coming from a single charge radiates outwards, so it is not surprising that its divergence is a constant times the charge we have.

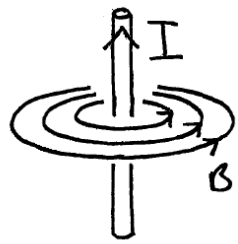
Gauss' law for magnetic fields:

$$\nabla \cdot \vec{B} = 0$$

This says the magnetic field equivalent, saying that there is no such thing as magnetic monopoles, as far as we know.

Ampère's law, original version:

We know of no such thing as magnetic monopoles, so how are magnetic fields generated? The original form of Ampère's law says that electric current generates magnetic field:



$$\nabla \times \vec{B} = \mu_0 J$$

where J is current density, and $\mu_0 = 4\pi \times 10^{-7} N/A^2$ (newtons per ampere squared) is the magnetic constant, or permeability of free space. A wire pointing straight up will have magnetic field going around it in clockwise direction.

Ampère's law with Maxwell's correction:

Maxwell's correction to Ampère's law says that changing electric field also produces magnetic field. We can consider a changing electric field as a kind of current, referred to as *displacement current*: $J_D = \epsilon_0 \frac{\partial \vec{E}}{\partial t}$. With this addition, Ampére's law reads:

$$\nabla \times \vec{B} = \mu_0 \left(J + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$$

Faraday's law:

Just like a changing electric field has a magnetic field associated with it, a changing magnetic field has an electric field associated with it. The equation says

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

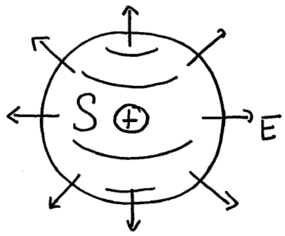
We shall discuss it some more when we get to integral form of Maxwell's equations.

16.7: Gauss' law, integral form, take I

The electric field for a single charge q at the origin was

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$$

Let's measure the flux of this electric field out of a sphere of radius R, centered at the origin. The coordinates are



$$r(\phi,\theta) = R(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi) = R\hat{r}$$

and

$$r_\phi \times r_\theta = R^2 \sin\phi\ \hat{r}$$

The electric field in these coordinates is

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}\hat{r}$$

The dot product of these two is

$$\vec{E}\cdot(r_\phi \times r_\theta) = \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}R^2\sin\phi\ |\hat{r}|^2 = \frac{q}{4\pi\epsilon_0}\sin\phi$$

The integral is

$$\iint_S \vec{E}\cdot d\vec{S} = \int_0^\pi \int_0^{2\pi} \frac{q}{4\pi\epsilon_0}\sin\phi\ d\theta d\phi = \frac{q}{2\epsilon_0}\ (-\cos\phi)|_0^\pi = \frac{q}{\epsilon_0}$$

This does not depend on the radius of the sphere. Or, as it turns out, on the surface being a sphere. We will see that later on, when we discuss the divergence theorem.

16.8: Ampère and Faraday's laws, integral form

Ampère's law:

The differential form of Ampère's law says

$$\nabla \times \vec{B} = \mu_0 \left(J + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$$

It says that an electric and displacement currents have a magnetic field associated with them. What is the total magnetic field rotating around?

Let us take a curve C, and a surface S with C as a boundary. We want to relate the magnetic field along C to the total current flowing through S. The differential form of Ampère's law allows us to do that, with the help of Stokes' theorem:

$$\oint_C \vec{B} \cdot d\vec{r} = \iint_S \left(\nabla \times \vec{B} \right) \cdot dS = \iint_S \left(\mu_0 J + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \cdot d\vec{S}$$

The first term of the right hand side integral simply gives the total electric current I through S. We can put that in, and get what's called the integral form of Ampère's law:

$$\oint_C \vec{B} \cdot d\vec{r} = \mu_0 I + \iint_S \left(\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \cdot d\vec{S}$$

Faraday's law:

The differential form of Faraday's law is

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

It says that a changing magnetic field has an electric field associated with it, rotating around. Faraday discovered that when a wire loop has a changing magnetic field going through it, then the wire loop acquires what's called an EMF (electromotive force) - a voltage, if you like, if one cut the loop open and measured it.

The integral form of Faraday's law describes this phenomenon: we shall get it from the differential form, using Stokes' theorem. The wire loop is the curve C, and S any surface with C as a boundary, oriented to match.

$$\oint_C \vec{E} \cdot d\vec{r} = \iint_S \left(\nabla \times \vec{E} \right) \cdot d\vec{S} = -\iint_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}$$

If C does not change with time, then d/dt can be pulled out of the right hand integral. We shall leave it in, stating the integral form of Faraday's law:

$$\oint_C \vec{E} \cdot d\vec{r} = -\iint_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}$$

16.9: Gauss' law, integral form, take II

The differential form of Gauss' law says

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

where ρ is charge density.

We have seen before that if we enclose a single charge q in the center of a sphere of any radius, and measure the flux of $\vec{E}$ (due to the charge) through the sphere, we get $\frac{q}{\epsilon_0}$. The divergence theorem lets us understand why that's the case, and extend this observation to any surface with any amount of charge inside it.

We take S to be any closed surface, oriented outward, enclosing a region V in space. To calculate the flux of $\vec{E}$ through S, allowing any amount of charge in our sphere. The divergence theorem says

$$\iint_S \vec{E} \cdot d\vec{S} = \iiint_V \left(\nabla \cdot \vec{E}\right) dV = \iiint_V \frac{\rho}{\epsilon_0}\, dV$$

Integrating charge density ρ over V gives the total charge Q enclosed by S. With that, Gauss' law in integral form says

$$\iint_S \vec{E} \cdot d\vec{S} = \frac{Q}{\epsilon_0}$$

The integral version of Gauss's law for magnetic fields is gotten exactly the same way, using the fact that there are no magnetic monopoles that we know of: the differential form is

$$\nabla \cdot \vec{B} = 0$$

which translates to

$$\iint_S \vec{B} \cdot d\vec{S} = 0$$

Maxwell's equations

	Differential form	Integral form
Gauss's law	$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$	$\iint_S \vec{E} \cdot d\vec{S} = \frac{Q}{\epsilon_0}$
Gauss's law for magnetism	$\nabla \cdot \vec{B} = 0$	$\iint_S \vec{B} \cdot d\vec{S} = 0$
Ampère's law	$\nabla \times \vec{B} = \mu_0 \left(J + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$	$\oint_C \vec{B} \cdot d\vec{r} = \mu_0 I + \iint_S \left(\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \cdot d\vec{S}$
Faraday's law	$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$	$\oint_C \vec{E} \cdot d\vec{r} = -\iint_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}$

Symbol chart

Symbol	Name	SI Units	in words	value (if applicable)
$\vec{E}$	Electric field	V/m	volt/meter	
$\vec{B}$	Magnetic field	T	tesla	
J	Current density	A/m^2	ampere / square meter	
I	Current	A	ampere	
ρ	Charge density	C	coulomb / cubic meter	
ϵ_0	Electric constant	F/m	farad / meter	$8.8541878 \times 10^{-12}$
μ_0	Magnetic constant	N/A^2	newton / ampere squared	$4\pi \times 10^{-7}$