On the supersymmetric XXX spin chain associated to $\mathfrak{gl}_{1|1}$

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Super Yangian $\mathrm{Y}(\mathfrak{gl}_{1|1})$

The **super Yangian** $\mathrm{Y}(\mathfrak{gl}_{1|1})$ is a unital associative superalgebra.

Generators: $T_{ij}^{(r)}$ of parity $|i|+|j|$, $i,j=1,2$ and $r\geqslant 1$;

Defining relations:

$$\mathcal{R}^{(1,2)}(x_1-x_2)T^{(1,3)}(x_1)T^{(2,3)}(x_2)=T^{(2,3)}(x_2)T^{(1,3)}(x_1)\mathcal{R}^{(1,2)}(x_1-x_2),$$

where $\mathcal{R}(x)=1+\dfrac{\mathcal{P}}{x}$ ($\mathcal{P}$ is the super flip operator) and

$$T(x)=\begin{pmatrix} T_{11}(x) & T_{12}(x) \\ T_{21}(x) & T_{22}(x)\end{pmatrix},\qquad T_{ij}(x)=\delta_{ij}+\sum_{r=1}^{\infty}T_{ij}^{(r)}x^{-r}.$$

Define the **Berezinian** $\mathrm{Ber}(x)$ [Nazarov91] by

$$\mathrm{Ber}(x)=T_{11}(x)(T_{22}(x)-T_{12}(x)T_{11}^{-1}(x)T_{21}(x))^{-1}.$$

The coefficients of Berezinian generate the **center** of $\mathrm{Y}(\mathfrak{gl}_{1|1})$ [Gow06].

Transfer matrices

In XXX spin chains, we study the spectrum of the action of **transfer matrices** on finite-dimensional irreducible $\mathrm{Y}(\mathfrak{gl}_{1|1})$-modules. We focus on the first transfer matrix.

The **(first) transfer matrix** $\mathcal{T}(x)$ is

$$\mathcal{T}(x) = \mathrm{str}(T(x)) = T_{11}(x) - T_{22}(x).$$

The transfer matrix $\mathcal{T}(x)$ satisfies

$$[\mathcal{T}(x_1), \mathcal{T}(x_2)] = 0 \quad \text{and} \quad [\mathcal{T}(x_1), T_{ij}^{(1)}] = 0.$$

Our goal is to

- find eigenvectors and corresponding eigenvalues of $\mathcal{T}(x)$;
- determine the dimension of each eigenspace;
- determine the size of each Jordan block.

Algebraic Bethe ansatz I

Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \cdots, \lambda^{(k)})$ be a sequence of pairs of complex numbers. Let $\boldsymbol{b} = (b_1, \cdots, b_k)$ be a sequence of complex numbers. Consider an irreducible tensor product of evaluation modules $\bigotimes_{s=1}^{k} V_{\lambda^{(s)}}(b_s)$. Then

$$\phi_{\boldsymbol{\lambda},\boldsymbol{b}}(x) = \prod_{s=1}^{k}(x - b_s + \lambda_1^{(s)}), \quad \psi_{\boldsymbol{\lambda},\boldsymbol{b}}(x) = \prod_{s=1}^{k}(x - b_s - \lambda_2^{(s)}).$$

are relatively prime [Zhang95].

Let $\boldsymbol{t} = (t_1, \cdots, t_l)$ be a sequence of complex numbers. Define the **off-shell Bethe vector** $\mathbb{B}_l(\boldsymbol{t})$ by

$$\mathbb{B}_l(\boldsymbol{t}) = \prod_{i=1}^{l}\prod_{s=1}^{k}(t_i - b_s) \prod_{1\leqslant i<j\leqslant l} \frac{1}{t_j - t_i + 1} T_{12}(t_1)\cdots T_{12}(t_l)\,|0\rangle.$$

Note that $\mathbb{B}_l(\boldsymbol{t})$ is symmetric in $\boldsymbol{t}$.

Algebraic Bethe ansatz II

Let $y(x) = (x - t_1) \cdots (x - t_l)$. If

$$y(x) \text{ divides } \phi_{\boldsymbol{\lambda},\boldsymbol{b}}(x) - \psi_{\boldsymbol{\lambda},\boldsymbol{b}}(x),$$

we call $\mathbb{B}_l(\boldsymbol{t})$ an **on-shell Bethe vector**.

Proposition [Kulish85, Belliard-Ragoucy09]

If an on-shell Bethe vector $\mathbb{B}_l(\boldsymbol{t})$ is nonzero, then $\mathbb{B}_l(\boldsymbol{t})$ is an eigenvector of $\mathcal{T}(x)$ with eigenvalue $\mathcal{E}_{\boldsymbol{\lambda},\boldsymbol{b},\boldsymbol{t}}(x) = \dfrac{\phi_{\boldsymbol{\lambda},\boldsymbol{b}}(x) - \psi_{\boldsymbol{\lambda},\boldsymbol{b}}(x)}{\prod_{s=1}^{k}(x - b_s)} \dfrac{y(x-1)}{y(x)}$.

Direct computation implies that

$$\mathcal{T}(x)\mathbb{B}_l(\boldsymbol{t}) = \mathcal{E}_{\boldsymbol{\lambda},\boldsymbol{b},\boldsymbol{t}}(x)\mathbb{B}_l(\boldsymbol{t}) + \sum_{i=1}^{l} \frac{\phi_{\boldsymbol{\lambda},\boldsymbol{b}}(t_i) - \psi_{\boldsymbol{\lambda},\boldsymbol{b}}(t_i)}{y'(t_i)} \frac{y(x-1)}{(x-t_i)(x-t_i-1)} \mathbb{B}_l(\boldsymbol{t}_i, x).$$

Similarly, one shows $e_{12}\mathbb{B}_l(\boldsymbol{t}) = 0$.

Completeness: Does this construction give all eigenvectors of $\mathcal{T}(x)$ in $\big(\bigotimes_{s=1}^{k} V_{\lambda^{(s)}}\big)^{\rm sing}$? **Generically**, it works well and is well-known.

Theorem [L-Mukhin]

Suppose all $\lambda^{(s)}$ are polynomial $\mathfrak{gl}_{1|1}$ weights and $\bigotimes_{s=1}^{k} V_{\lambda^{(s)}}(b_s)$ is cyclic. Then

- Each eigenspace in $(\bigotimes_{s=1}^{k} V_{\lambda^{(s)}}(b_s))^{\rm sing}$ of $\mathcal{T}(x)$ is 1-dimensional.
- Eigenspaces of $\mathcal{T}(x)$ bijectively correspond to monic divisors y of $\phi_{\boldsymbol{\lambda},\boldsymbol{b}} - \psi_{\boldsymbol{\lambda},\boldsymbol{b}}$.
- The size of Jordan block corresponding to y is
$$\prod_{a\in\mathbb{C}} \binom{\mathrm{Mult}_a(\phi_{\boldsymbol{\lambda},\boldsymbol{b}} - \psi_{\boldsymbol{\lambda},\boldsymbol{b}})}{\mathrm{Mult}_a(y)},$$
where $\mathrm{Mult}_a(f)$ is the multiplicity of a as a root of f.
- If $\bigotimes_{s=1}^{k} V_{\lambda^{(s)}}(b_s)$ is irreducible, then all on-shell Bethe vectors are nonzero.

Higher transfer matrices

Theorem [L-Mukhin]

We have

$$\prod_{i=1}^{n} \mathcal{T}(x-i+1)$$

$$= \mathrm{str}(A_n T^{(1)}(x) T^{(2)}(x-1) \cdots T^{(n)}(x-n+1)) \prod_{i=1}^{n-1} (1 - \mathrm{Ber}(x-i))$$

$$= \mathrm{str}(H_n T^{(1)}(x) T^{(2)}(x-1) \cdots T^{(n)}(x-n+1)) \prod_{i=1}^{n-1} (1 - \mathrm{Ber}^{-1}(x-i))$$

Hence higher transfer matrices can be "expressed" in terms of the first transfer matrix and the center.

These formulas can be understood as follows.

We have the following equality in **Grothendieck ring**,

$$V_{n\omega_1}(x) \otimes V_{\omega_1}(x-n) = V_{(n+1)\omega_1}(x) + \big(\mathbb{C}^{\mathrm{odd}}_{(-1,1)}(x-n) \otimes V_{(n+1)\omega_1}(x)\big).$$

Inductively, we have the equality in Grothendieck ring,

$$\bigotimes_{i=1}^{n} V_{\omega_1}(x-i+1) = \sum_{\ell=0}^{n-1} \sum_{1\leqslant i_1<\cdots<i_\ell\leqslant n-1} \mathbb{C}_{i_1,\cdots,i_\ell} \otimes V_{n\omega_1}(x),$$

where

$$\mathbb{C}_{i_1,\cdots,i_\ell} = \bigotimes_{j=1}^{\ell} \mathbb{C}^{\mathrm{odd}}_{(-1,1)}(x-i_j).$$

Thank you!