TWISTED SUPER YANGIANS OF TYPE AIII AND THEIR REPRESENTATIONS

KANG LU

Abstract. We study the super analogue of the Molev-Ragoucy reflection algebras, which we call twisted super Yangians of type AIII, and classify their finite-dimensional irreducible representations. These superalgebras are coideal subalgebras of the super Yangian $\mathcal{Y}(\mathfrak{gl}_{m|n})$ and are associated with symmetric pairs of type AIII in Cartan's classification. We establish the Schur-Weyl type duality between degenerate affine Hecke algebras of type BC and twisted super Yangians.

Keywords: Schur-Weyl duality, twisted super Yangian, quantum symmetric pair

Contents

1. Introduction

Reflection algebras, introduced by Sklyanin in his seminal paper [Skl88], are pivotal in constructing the commutative Bethe subalgebra and ensuring integrability of quantum integrable systems with boundary conditions. These algebras, inspired by Cherednik's scattering theory [Che84] for factorized particles on the half-line, form the foundation for various studies.

In [MR02], Molev and Ragoucy studied a family of reflection algebras $\mathcal{B}_{\boldsymbol{\varepsilon}}$, whose relations are described in terms of reflection equation and a certain unitary condition, and classified their finite-dimensional irreducible representations. These reflection algebras can also be called twisted Yangians of type AIII as they are coideal subalgebras of the Yangian $\mathcal{Y}(\mathfrak{gl}_n)$ and deformations of the fixed point subalgebra of $\mathrm{U}(\mathfrak{gl}_n[x])$ associated to symmetric pair of type AIII, see §3.2. The twisted Yangians depend on a sequence $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)$, where $\varepsilon_i = \pm 1$, and for different $\boldsymbol{\varepsilon}$ the $\mathcal{B}_{\boldsymbol{\varepsilon}}$ might not be isomorphic.

These twisted Yangians were further investigated by Chen, Guay and Ma in [CGM14]. They related the twisted Yangians (in R-matrix presentation) with another family of twisted Yangians introduced by MacKay [Mac02] (in Drinfeld's original presentation). A Drinfeld functor from the category of modules over degenerate affine Hecke algebras of type BC (dAHA) to the category of modules over twisted Yangians were constructed. It turns out the Drinfeld functor is an equivalence of categories under certain conditions, similar to the usual Schur-Weyl duality. Moreover, the Drinfeld functor sends a finite-dimensional irreducible module over dAHA to either zero space or a finite-dimensional irreducible module over twisted Yangians.

In the present article, we shall study the supersymmetric generalization of $\mathcal{B}_{\boldsymbol{\varepsilon}}$, that are twisted super Yangians of type AIII. The twisted super Yangians $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$ are coideal subalgebras of the super Yangian

$\mathcal{Y}(\mathfrak{gl}^{\boldsymbol{s}}_{m|n})$ that depends on sequences of parity sequence $\boldsymbol{s} = (s_1, s_2, \cdots, s_{m+n})$ and $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{m+n})$, where $s_i, \varepsilon_i = \pm 1$ for $1 \leqslant i \leqslant m+n$. The new sequence $\boldsymbol{s}$ corresponds to the Dynkin diagram we choose for the associated general linear Lie superalgebra $\mathfrak{gl}_{m|n}$. When $\boldsymbol{s}$ satisfies $s_i = 1$ for $1 \leqslant i \leqslant m$ and $s_i = -1$ otherwise, we call $\boldsymbol{s}$ the standard parity sequence. The standard parity sequence corresponds to the standard Borel subalgebra of $\mathfrak{gl}_{m|n}$.

The twisted super Yangians appear previously (under the name *reflection superalgebras*) in the study of analytical and nested algebraic Bethe ansatz [RS07, BR09] for quantum integrable models (open spin chains) with symmetry described by twisted super Yangians. For the case of the standard parity sequence $\boldsymbol{s}$ and a specific $\boldsymbol{\varepsilon}$, they computed the highest weight of twisted super Yangian for a highest weight vector of super Yangians. They are also recently studied in [Ket23], where some partial results of this paper were obtained, and in [BK23], where a double version of twisted super Yangian is introduced and studied. Note that in [Ket23, BK23], the author deals with twisted super Yangians associated with the standard parity sequence $\boldsymbol{s}$ and a specific $\boldsymbol{\varepsilon}$ while ours are arbitrary[1].

Our primary objective is to obtain analogous results to [MR02, CGM14] with arbitrary $\boldsymbol{s}$ and $\boldsymbol{\varepsilon}$. We use similar strategy as in [MR02, CGM14]. Under our setting, $\boldsymbol{s}$ and $\boldsymbol{\varepsilon}$ are both arbitrary. The calculations become more complicated than that in [MR02, CGM14]. We need to put extra effort to correctly insert the necessary sign factors $\boldsymbol{s}$ and $\boldsymbol{\varepsilon}$.

Finite-dimensional irreducible representations of super Yangians were classified by Zhang [Zha95, Zha96] for the standard parity sequence. A complete and concrete description of criteria for an irreducible $\mathcal{Y}(\mathfrak{gl}^{\boldsymbol{s}}_{m|n})$-module (for arbitrary $\boldsymbol{s}$) being finite-dimensional is not available, though such a criteria can be obtained recursively using the odd reflections [Mol22, Lu22]. Consequently, we only have classification of finite-dimensional irreducible $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-modules for the cases (1) arbitrary $\boldsymbol{\varepsilon}$ when $n = 0, 1$ and (2) the standard parity sequence $\boldsymbol{s}$ when the occurrence of i such that $\varepsilon_i \neq \varepsilon_{i+1}$ is at most 1.

There are also twisted Yangians of types AI and AII introduced by Olshanski [Ols92] and of types BCD introduced by Guay and Regelskis [GR16] via R-matrix presentation. Another family of twisted Yangians associated to general symmetric pairs were introduced by MacKay [Mac02] in terms of Drinfeld's J-symbols. More recently, together with Wang and Zhang, we introduced another family of twisted Yangians for symmetric pairs of split types in Drinfeld's new presentation, [LWZ23, LWZ24]. The isomorphism between these families remains unproven, offering an interesting avenue for future research. Results for certain types like type AI and AIII can be found in [LWZ23, CGM14], respectively. It is an interesting question to find Drinfeld's original and new presentations for twisted super Yangians of type AIII.

This article is organized in the following fashion. Section 2 revisits basic properties of the super Yangian $\mathcal{Y}(\mathfrak{gl}^{\boldsymbol{s}}_{m|n})$. Section 3 delves into twisted super Yangians and their properties. Section 4 explores highest weight representation theory and tensor product structures for twisted super Yangians. Section 5 classifies finite-dimensional irreducible representations for rank 1, while Section 6 extends this classification to higher ranks in key cases. Finally, Section 7 establishes a Schur-Weyl type duality between degenerate affine Hecke algebras of type BC and twisted super Yangians.

Acknowledgments. The author is partially supported by Weiqiang Wang's NSF grant DMS-2001351.

2. Super Yangian

sec:superyangian

2.1. **General linear Lie superalgebras.** Throughout the paper, we work over $\mathbb{C}$. In this section, we recall the basics of the general linear Lie superalgebra $\mathfrak{gl}^{\boldsymbol{s}}_{m|n}$, see e.g. [CW12] for more detail.

A *vector superspace* $W = W_{\bar{0}} \oplus W_{\bar{1}}$ is a $\mathbb{Z}_2$-graded vector space. We call elements of $W_{\bar{0}}$ *even* and elements of $W_{\bar{1}}$ *odd*. We write $|w| \in \{\bar{0}, \bar{1}\}$ for the parity of a homogeneous element $w \in W$. Set $(-1)^{\bar{0}} = 1$ and $(-1)^{\bar{1}} = -1$.

[1]For the classical limit, the treatment is the same for twisted super Yangians associated to different $\boldsymbol{s}$ and $\boldsymbol{\varepsilon}$, but the representations theory does rely on $\boldsymbol{s}$ and $\boldsymbol{\varepsilon}$.

Fix $m,n\in\mathbb{Z}_{\geqslant 0}$ and set $\varkappa=m+n$. Denote by $S_{m|n}$ the set of all sequences $\boldsymbol{s}=(s_1,s_2,\dots,s_\varkappa)$ where $s_i\in\{\pm1\}$ and 1 occurs exactly m times. Elements of $S_{m|n}$ are called *parity sequences*. The parity sequence of the form $\boldsymbol{s_0}=(1,\dots,1,-1,\dots,-1)$ is the *standard parity sequence*.

Fix a parity sequence $\boldsymbol{s}\in S_{m|n}$ and define $|i|\in\mathbb{Z}_2$ for $1\leqslant i\leqslant\varkappa$ by $s_i=(-1)^{|i|}$.

The Lie superalgebra $\mathfrak{gl}^{\boldsymbol{s}}_{m|n}$ is generated by elements $e^{\boldsymbol{s}}_{ij}$, $1\leqslant i,j\leqslant\varkappa$, with the supercommutator relations

$$[e^{\boldsymbol{s}}_{ij},e^{\boldsymbol{s}}_{kl}]=\delta_{jk}e^{\boldsymbol{s}}_{il}-(-1)^{(|i|+|j|)(|k|+|l|)}\delta_{il}e^{\boldsymbol{s}}_{kj},$$

where the parity of $e^{\boldsymbol{s}}_{ij}$ is $|i|+|j|$. In the following, we shall drop the superscript $\boldsymbol{s}$ when there is no confusion.

Denote by $\mathrm{U}(\mathfrak{gl}^{\boldsymbol{s}}_{m|n})$ the universal enveloping superalgebra of $\mathfrak{gl}^{\boldsymbol{s}}_{m|n}$. The superalgebra $\mathrm{U}(\mathfrak{gl}^{\boldsymbol{s}}_{m|n})$ is a Hopf superalgebra with the coproduct given by $\Delta(x)=1\otimes x+x\otimes 1$ for all $x\in\mathfrak{gl}^{\boldsymbol{s}}_{m|n}$.

The *Cartan subalgebra* $\mathfrak{h}$ of $\mathfrak{gl}^{\boldsymbol{s}}_{m|n}$ is spanned by e_{ii}, $1\leqslant i\leqslant\varkappa$. Let ϵ_i, $1\leqslant i\leqslant\varkappa$, be a basis of $\mathfrak{h}^*$ (the dual space of $\mathfrak{h}$) such that $\epsilon_i(e_{jj})=\delta_{ij}$. There is a bilinear form $(\ ,\ )$ on $\mathfrak{h}^*$ given by $(\epsilon_i,\epsilon_j)=s_i\delta_{ij}$. The *root system* $\boldsymbol{\Phi}$ is a subset of $\mathfrak{h}^*$ given by

$$\boldsymbol{\Phi}:=\{\epsilon_i-\epsilon_j\mid 1\leqslant i,j\leqslant\varkappa\text{ and }i\neq j\}.$$

We call a root $\epsilon_i-\epsilon_j$ *even* (resp. *odd*) if $|i|=|j|$ (resp. $|i|\neq|j|$).

Set $\alpha_i:=\epsilon_i-\epsilon_{i+1}$ for $1\leqslant i\leqslant\varkappa$. Denote by

$$\mathbf{P}:=\bigoplus_{1\leqslant i\leqslant\varkappa}\mathbb{Z}\epsilon_i,\quad \mathbf{Q}:=\bigoplus_{1\leqslant i<\varkappa}\mathbb{Z}\alpha_i,\quad \mathbf{Q}_{\geqslant 0}:=\bigoplus_{1\leqslant i<\varkappa}\mathbb{Z}_{\geqslant 0}\alpha_i$$

the *weight lattice*, the *root lattice*, and the *cone of positive roots*, respectively. Define a partial ordering $\geqslant$ on $\mathfrak{h}^*$: $\mu\geqslant\nu$ if $\mu-\nu\in\mathbf{Q}_{\geqslant 0}$.

A module M over a superalgebra $\mathscr{A}$ is a vector superspace M with a homomorphism of superalgebras $\mathscr{A}\to\mathrm{End}(M)$. A $\mathfrak{gl}^{\boldsymbol{s}}_{m|n}$-module is a module over $\mathrm{U}(\mathfrak{gl}^{\boldsymbol{s}}_{m|n})$. However, we shall not distinguish modules which only differ by a parity.

For a $\mathfrak{gl}^{\boldsymbol{s}}_{m|n}$-module M, define the *weight subspace of weight* μ by

eq:uweight-space

$$(M)_\mu:=\{v\in M\mid e_{ii}v=\mu(e_{ii})v,\ 1\leqslant i\leqslant\varkappa\}. \tag{2.1}$$

For a $\mathfrak{gl}^{\boldsymbol{s}}_{m|n}$-module M such that $(M)_\mu=0$ unless $\mu\in\mathbf{Q}$, we say that M is $\mathbf{Q}$-*graded*.

For a $\mathfrak{gl}^{\boldsymbol{s}}_{m|n}$-module M, we call a vector $v\in M$ *singular* if $e_{ij}v=0$ for $1\leqslant i<j\leqslant\varkappa$. We call a nonzero vector $v\in M$ a *singular vector of weight* μ if v satisfies

$$e_{ii}v=\mu(e_{ii})v,\quad e_{jk}v=0,$$

for $1\leqslant i\leqslant\varkappa$ and $1\leqslant j<k\leqslant\varkappa$. A nonzero vector $v\in(M)_\mu$ is *a highest (resp. lowest) weight vector* of M if $(M)_\nu=0$ unless $\mu-\nu\in\mathbf{Q}_{\geqslant\mathbf{0}}$ (resp. $\nu-\mu\in\mathbf{Q}_{\geqslant\mathbf{0}}$). Clearly, a highest weight vector is singular while a lowest weight vector v satisfies $e_{ji}v=0$ for $1\leqslant i<j\leqslant\varkappa$.

Denote by $L(\mu)$ the irreducible $\mathfrak{gl}^{\boldsymbol{s}}_{m|n}$-module generated by a singular vector of weight μ.

Let $V:=\mathbb{C}^{m|n}$ be the vector superspace with a basis v_i, $1\leqslant i\leqslant\varkappa$, such that $|v_i|=|i|$. Let $E_{ij}\in\mathrm{End}(V)$ be the linear operators such that $E_{ij}v_k=\delta_{jk}v_i$. The map $\rho_V:\mathfrak{gl}^{\boldsymbol{s}}_{m|n}\to\mathrm{End}(V)$, $e_{ij}\mapsto E_{ij}$ defines a $\mathfrak{gl}^{\boldsymbol{s}}_{m|n}$-module structure on V. As a $\mathfrak{gl}^{\boldsymbol{s}}_{m|n}$-module, V is isomorphic to $L(\epsilon_1)$. The vector v_i has weight ϵ_i. The highest weight vector is v_1 and the lowest weight vector is v_{m+n}. We call it the *vector representation* of $\mathfrak{gl}^{\boldsymbol{s}}_{m|n}$.

sec rtt

2.2. Super Yangians.

Fix a parity sequence $\boldsymbol{s}\in S_{m|n}$ and recall the definition of super Yangian $\mathscr{Y}_{\boldsymbol{s}}:=\mathscr{Y}(\mathfrak{gl}^{\boldsymbol{s}}_{m|n})$ from [Naz91].

Definition 2.1. *The super Yangian $\mathscr{Y}_{\boldsymbol{s}}$ is the $\mathbb{Z}_2$-graded unital associative algebra over $\mathbb{C}$ with generators $\{t_{ij}^{(r)} \mid 1 \leqslant i,j \leqslant \varkappa,\ r \geqslant 1\}$ and the defining relations are given by*

eq:comm-series
$$[t_{ij}(u), t_{kl}(v)] = \frac{(-1)^{|i||j|+|i||k|+|j||k|}}{u-v}(t_{kj}(u)t_{il}(v) - t_{kj}(v)t_{il}(u)). \tag{2.2}$$

where

$$t_{ij}(u) = \sum_{k=0}^{\infty} t_{ij}^{(k)} u^{-k}, \qquad t_{ij}^{(0)} = \delta_{ij},$$

and the generators $t_{ij}^{(r)}$ have parities $|i| + |j|$.

The super Yangian $\mathscr{Y}_{\boldsymbol{s}}$ has the RTT presentation as follows. Define the rational R-matrix $R(u) \in \mathrm{End}(V \otimes V)$ by $R(u) = 1 - \mathscr{P}/u$, where $\mathscr{P} \in \mathrm{End}(V \otimes V)$ is the super flip operator defined by

$$\mathscr{P} = \sum_{i,j=1}^{\varkappa} s_j E_{ij} \otimes E_{ji}.$$

The rational R-matrix satisfies the quantum Yang-Baxter equation

eq yang-baxter
$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v). \tag{2.3}$$

Define the operator $T(u) \in \mathscr{Y}_{\boldsymbol{s}}[[u^{-1}]] \otimes \mathrm{End}(V)$,

$$T(u) = \sum_{i,j=1}^{\varkappa} (-1)^{|i||j|+|j|} t_{ij}(u) \otimes E_{ij}.$$

Then defining relations (2.2) can be written as

eq:RTT
$$R(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u-v) \in \mathscr{Y}_{\boldsymbol{s}}[[u^{-1}]] \otimes \mathrm{End}(V^{\otimes 2}). \tag{2.4}$$

The super Yangian $\mathscr{Y}_{\boldsymbol{s}}$ is a Hopf superalgebra with the coproduct

eq Hopf
$$\Delta : t_{ij}(u) \mapsto \sum_{k=1}^{\varkappa} t_{ik}(u) \otimes t_{kj}(u), \tag{2.5}$$

and the antipode $S : T(u) \to T^{-1}(u)$.

Define the series

$$t'_{ij}(u) = \sum_{k=0}^{\infty} t'^{(k)}_{ij} u^{-k}$$

by

eq:inverseT
$$T^{-1}(u) = \sum_{i,j=1}^{\varkappa} (-1)^{|i||j|+|j|} t'_{ij}(u) \otimes E_{ij}. \tag{2.6}$$

Then

eq:T'-expression
$$t'_{ij}(u) = \delta_{ij} + \sum_{k=1}^{\infty} (-1)^k \sum_{a_1,\cdots,a_{k-1}=1}^{\varkappa} t^{\circ}_{ia_1}(u) t^{\circ}_{a_1a_2}(u) \cdots t^{\circ}_{a_{k-1}j}(u), \tag{2.7}$$

where $t^{\circ}_{ij}(u) = t_{ij}(u) - \delta_{ij}$. In particular, by taking the coefficient of u^{-r}, for $r \geqslant 1$, one obtains

eq:T-expression-comp
$$t'^{(r)}_{ij} = \sum_{k=1}^{r} (-1)^k \sum_{a_1,\cdots,a_{k-1}=1}^{\varkappa} \sum_{r_1+\cdots+r_k=r} t^{(r_1)}_{ia_1} t^{(r_2)}_{a_1a_2} \cdots t^{(r_k)}_{a_{k-1}j}, \tag{2.8}$$

where r_i for $1 \leqslant i \leqslant k$ are positive integers.

By (2.4), one has

eq:T'RT
$$\begin{aligned} T_1^{-1}(-u)R(u+v)T_2(v) &= T_2(v)R(u+v)T_1^{-1}(-u), \\ T_1(u)R(u+v)T_2^{-1}(-v) &= T_2^{-1}(-v)R(u+v)T_1(u), \end{aligned} \tag{2.9}$$

and

eq:tt'
$$(u-v)[t_{ij}(u),t'_{kl}(v)]=(-1)^{|i||j|+|i||k|+|j||k|}\Big(\delta_{kj}\sum_{s=1}^{\varkappa}t_{is}(u)t'_{sl}(v)-\delta_{il}\sum_{s=1}^{\varkappa}t'_{ks}(v)t_{sj}(u)\Big). \tag{2.10}$$

For $z\in\mathbb{C}$ there exists an isomorphism of Hopf superalgebras,

eq tau z
$$\tau_z:\mathscr{Y}_{\boldsymbol{s}}\to\mathscr{Y}_{\boldsymbol{s}},\qquad t_{ij}(u)\mapsto t_{ij}(u-z). \tag{2.11}$$

The universal enveloping superalgebra $\mathrm{U}(\mathfrak{gl}^{\boldsymbol{s}}_{m|n})$ is a Hopf subalgebra of $\mathscr{Y}_{\boldsymbol{s}}$ via the embedding $e_{ij}\mapsto s_it_{ij}^{(1)}$. The left inverse of this embedding is the *evaluation homomorphism* $\pi^{\boldsymbol{s}}_{m|n}:\mathscr{Y}_{\boldsymbol{s}}\to\mathrm{U}(\mathfrak{gl}^{\boldsymbol{s}}_{m|n})$ given by

eq:evaluation-map
$$\pi^{\boldsymbol{s}}_{m|n}:t_{ij}(u)\mapsto\delta_{ij}+s_ie_{ij}u^{-1}. \tag{2.12}$$

The evaluation homomorphism is a superalgebra homomorphism but not a Hopf superalgebra homomorphism. For any $\mathfrak{gl}^{\boldsymbol{s}}_{m|n}$-module M, it is naturally a $\mathscr{Y}_{\boldsymbol{s}}$-module obtained by pulling back M through the evaluation homomorphism $\pi_{m|n}$. We denote the corresponding $\mathscr{Y}_{\boldsymbol{s}}$-module by the same letter M and call it an *evaluation module*.

The following standard PBW-type theorem for super Yangian $\mathscr{Y}_{\boldsymbol{s}}$ is known.

thm:PBW

Theorem 2.2 ([Gow07, Pen16]). *Given any total ordering on the elements $t_{ij}^{(p)}$ for $1\leqslant i,j\leqslant\varkappa$ and $p\in\mathbb{Z}_{>0}$, the ordered monomials in these elements, containing no second or higher order powers of the odd generators, form a basis of the super Yangian $\mathscr{Y}_{\boldsymbol{s}}$.*

Besides the antipode S, we also have the following anti-automorphisms of $\mathscr{Y}_{\boldsymbol{s}}$ defined by

$$\begin{aligned}&\boldsymbol{t}:\mathscr{Y}_{\boldsymbol{s}}\to\mathscr{Y}_{\boldsymbol{s}},\quad t_{ij}(u)\mapsto(-1)^{|i||j|+|j|}t_{ji}(u),\\&\boldsymbol{n}:\mathscr{Y}_{\boldsymbol{s}}\to\mathscr{Y}_{\boldsymbol{s}},\quad t_{ij}(u)\mapsto t_{ij}(-u).\end{aligned}$$

Then the anti-automorphisms S, $\boldsymbol{t}$, and $\boldsymbol{n}$ of $\mathscr{Y}_{\boldsymbol{s}}$ pairwise commute, see e.g. [Naz20, Proposition 1.5]. Let Ω be the anti-automorphism of $\mathscr{Y}_{\boldsymbol{s}}$ given by

Omega
$$\Omega=S\circ\boldsymbol{t}\circ\boldsymbol{n},\qquad\Omega(t_{ij}(u))=(-1)^{|i||j|+|j|}t'_{ji}(-u). \tag{2.13}$$

2.3. **Highest weight representations.** We first recall the results about the highest weight representations for $\mathscr{Y}_{\boldsymbol{s}}$ from [Zha96].

Definition 2.3. *A representation L of $\mathscr{Y}_{\boldsymbol{s}}$ is called highest $\ell_{\boldsymbol{s}}$-weight if there exists a nonzero vector $\xi\in L$ such that L is generated by ξ and ξ satisfies*

eq:highest-Y
$$\begin{aligned}&t_{ij}(u)\xi=0,&&1\leqslant i<j\leqslant\varkappa,\\&t_{ii}(u)\xi=\lambda_i(u)\xi,&&1\leqslant i\leqslant\varkappa,\end{aligned} \tag{2.14}$$

where $\lambda_i(u)\in 1+u^{-1}\mathbb{C}[[u^{-1}]]$. The vector ξ is called a highest $\ell_{\boldsymbol{s}}$-weight vector of L and the tuple $\boldsymbol{\lambda}(u)=(\lambda_i(u))_{1\leqslant i\leqslant\varkappa}$ is the highest $\ell_{\boldsymbol{s}}$-weight of L.

Let $\boldsymbol{\lambda}(u)=(\lambda_i(u))_{1\leqslant i\leqslant\varkappa}$ be a $\varkappa$-tuple as above. Then there exists a unique, up to isomorphism, irreducible highest weight representation $L(\boldsymbol{\lambda}(u))$ with the highest weight $\boldsymbol{\lambda}(u)$. Any finite-dimensional irreducible representation of $\mathscr{Y}_{\boldsymbol{s}}$ is isomorphic to $L(\boldsymbol{\lambda}(u))$ for some $\boldsymbol{\lambda}(u)$. The criterion for $L(\boldsymbol{\lambda}(u))$ being finite-dimensional was classified in [Zha96] when $\boldsymbol{s}$ is the standard parity sequence.

thm:zhang

Theorem 2.4 ([Zha96]). *If $\boldsymbol{s}$ is the standard parity sequence, then the irreducible $\mathscr{Y}_{\boldsymbol{s}}$-module $L(\boldsymbol{\lambda}(u))$ is finite-dimensional if and only if there exist monic polynomials $P_i(u)$, $1\leqslant i\leqslant\varkappa$, such that*

$$\frac{\lambda_i(u)}{\lambda_{i+1}(u)}=\frac{P_i(u+s_i)}{P_i(u)},\qquad\frac{\lambda_m(u)}{\lambda_{m+1}(u)}=\frac{P_m(u)}{P_\varkappa(u)},\quad 1\leqslant i\leqslant\varkappa\ \textit{and}\ i\neq m,$$

and $\deg P_m=\deg P_\varkappa$.

A criterion for an arbitrary parity sequence $\boldsymbol{s}$ can be recursively deduced from Theorem 2.4 via the odd reflections of super Yangian, see [Mol22, Lu22]. However, a compact description of such a criterion for an arbitrary parity sequence $\boldsymbol{s}$ is not available.

Regard $\mathrm{U}(\mathfrak{gl}^{\boldsymbol{s}}_{m|n})$ as a subalgebra of $\mathscr{Y}_{\boldsymbol{s}}$, then we have $t_{ii}^{(1)} = s_i e_{ii}$. In particular, one assigns a $\mathfrak{gl}^{\boldsymbol{s}}_{m|n}$-weight to an $\ell_{\boldsymbol{s}}$-weight via the map

varpi1
$$\varpi : \mathfrak{B} \to \mathfrak{h}^*,\ \boldsymbol{\lambda}(u) \mapsto \varpi(\boldsymbol{\lambda}(u)) \quad \text{such that} \quad \varpi(\boldsymbol{\lambda}(u))(e_{ii}) = s_i\lambda_{i,1}, \tag{2.15}$$

where $\lambda_{i,1}$ is the coefficients of u^{-1} in $\lambda_i(u)$.

Given a $\mathscr{Y}_{\boldsymbol{s}}$-module L, consider it as a $\mathfrak{gl}^{\boldsymbol{s}}_{m|n}$-module and its $\mathfrak{gl}^{\boldsymbol{s}}_{m|n}$-weight subspaces $(M)_\mu$, see (2.1).

lem:wt-change

Lemma 2.5. *We have*

$$t_{ij}^{(r)}(M)_\mu \subset (M)_{\mu+\epsilon_i-\epsilon_j}, \qquad t_{ij}'^{(r)}(M)_\mu \subset (M)_{\mu+\epsilon_i-\epsilon_j},$$

for $1 \leqslant i,j \leqslant \varkappa$, *and* $r \in \mathbb{Z}_{>0}$.

Proof. By (2.2) and (2.10), we have

eq:1st-node-tt
$$[t_{ij}^{(1)}, t_{kl}(u)] = (-1)^{|i||j|+|i||k|+|j||k|}\big(\delta_{kj}t_{il}(u) - \delta_{il}t_{kj}(u)\big), \tag{2.16}$$

eq:1st-node-tt'
$$[t_{ij}^{(1)}, t'_{kl}(u)] = (-1)^{|i||j|+|i||k|+|j||k|}\big(\delta_{kj}t'_{il}(u) - \delta_{il}t'_{kj}(u)\big). \tag{2.17}$$

Note that $t_{ij}^{(1)}$ is identified with $s_i e_{ij}$, then the lemma follows from the above equations by a direct computation. □

Thus, we have the following corollary of Theorem 2.2 and Lemma 2.5.

Corollary 2.6. *If L is a $\mathscr{Y}_{\boldsymbol{s}}$-module of highest $\ell_{\boldsymbol{s}}$-weight $\boldsymbol{\lambda}(u)$, then L has a $\mathfrak{gl}^{\boldsymbol{s}}_{m|n}$-weight subspace decomposition. Moreover, its highest $\ell_{\boldsymbol{s}}$-weight vector has weight $\varpi(\boldsymbol{\lambda}(u))$ and the other weight vectors have weights that are strictly smaller (with respect to $\geqslant_{\boldsymbol{s}}$ defined in §2.1) than $\varpi(\boldsymbol{\lambda}(u))$.* □

Let $\mathscr{Y}_{\boldsymbol{s}}^{+}$ be the left ideal of $\mathscr{Y}_{\boldsymbol{s}}$ generated by all the coefficients of $t_{ij}(u)$ with $1 \leqslant i < j \leqslant \varkappa$. We write $X \doteq X'$ if $X - X' \in \mathscr{Y}_{\boldsymbol{s}}^{+}$. Clearly, if ξ is a highest $\ell_{\boldsymbol{s}}$-weight vector of $\mathscr{Y}_{\boldsymbol{s}}$ and $X \doteq X'$, then $X\xi = X'\xi$.

prop:t'-l-weight

Proposition 2.7 ([RS07, BR09])**.** *If ξ is a highest $\ell_{\boldsymbol{s}}$-weight vector of highest $\ell_{\boldsymbol{s}}$-weight $\boldsymbol{\lambda}(u)$ in a representation L of $\mathscr{Y}_{\boldsymbol{s}}$, then*

eq:t'-annih
$$\begin{aligned} t'_{ij}(u)\xi &= 0, && 1 \leqslant i < j \leqslant \varkappa,\\ t'_{ii}(u)\xi &= \lambda'_i(u)\xi, && 1 \leqslant i \leqslant \varkappa, \end{aligned} \tag{2.18}$$

for certain $\lambda'_i(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$. (The formal series $\lambda'_i(u)$ will be determined later.)

Proof. Let $1 \leqslant i < j \leqslant \varkappa$. By (2.10), for any $1 \leqslant k \leqslant \varkappa$, we have

$$(-1)^{|i||j|+|i||k|+|j||k|}[t_{kj}(u), t'_{ik}(v)] \doteq -\frac{1}{u-v}\sum_{s=j}^{\varkappa} t'_{is}(v)t_{sj}(u).$$

Expanding $(u-v)^{-1}$ as $\sum_{r=0}^{\infty} v^r u^{-r-1}$ and take the coefficients of $u^{-1}v^{-p}$ and $u^{-2}v^{-p}$, we have

prop:kill-1
$$(-1)^{|i||j|+|i||k|+|j||k|}[t_{kj}^{(1)}, t_{ik}'^{(p)}] \doteq -t_{ij}'^{(p)}, \tag{2.19}$$

prop:kill-2
$$(-1)^{|i||j|+|i||k|+|j||k|}[t_{kj}^{(2)}, t_{ik}'^{(p)}] \doteq -t_{ij}'^{(p+1)} - t_{ij}'^{(p)}t_{jj}^{(1)} - \sum_{s=j+1}^{\varkappa} t_{is}'^{(p)}t_{sj}^{(1)}. \tag{2.20}$$

We prove $t_{ij}'^{(p)}\xi = 0$ for all $1 \leqslant i < j \leqslant \varkappa$ by induction on p.

The base case is clear because it is immediate from (2.8) that $t'^{(1)}_{ij} = -t^{(1)}_{ij}$. Suppose now that $t'^{(p)}_{ij}\xi = 0$. It follows from (2.19) and the induction hypothesis that

prop:kill-3
$$t'^{(p)}_{is} t^{(1)}_{sj} \xi = 0, \qquad j < s \leqslant \varkappa. \tag{2.21}$$

Note that ξ is an eigenvector $t^{(1)}_{jj}$ and $t^{(2)}_{jj}$, we have

prop:kill-4
$$[t^{(2)}_{jj}, t'^{(p)}_{ij}]\xi = 0, \qquad t'^{(p)}_{ij} t^{(1)}_{jj} \xi = 0. \tag{2.22}$$

Setting $k = j$ in (2.20) and applying (2.20) to ξ, we immediately obtain $t'^{(p+1)}_{ij}\xi = 0$ from (2.21) and (2.22). Thus by induction, we have $t^{(r)}_{ij}\xi = 0$ for all $1 \leqslant i < j \leqslant \varkappa$ and $r \in \mathbb{Z}_{>0}$.

Since $\mathfrak{gl}^{\boldsymbol{s}}_{m|n}$ can be regarded as a subalgebra of $\mathscr{Y}_{\boldsymbol{s}}$, the $\mathscr{Y}_{\boldsymbol{s}}$-module L is hence a $\mathfrak{gl}^{\boldsymbol{s}}_{m|n}$-module and has the weight decomposition. The vector ξ has the weight $\varpi(\boldsymbol{\lambda})$. By Theorem 2.2 and Lemma 2.5, $(L)_{\varpi(\boldsymbol{\lambda})}$ is of dimension 1 and all other weights appearing in L are smaller than $\varpi(\boldsymbol{\lambda})$. It follows from Lemma 2.5 that $t'_{jj}(u)$ preserves $(L)_{\varpi(\boldsymbol{\lambda})}$ and hence preserves ξ. Therefore, $t'_{jj}(u)\xi = \lambda'_i(u)\xi$ for some $\lambda'_i(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$. □

By the same strategy, we have the following lemma.

lem:tia-kill

Lemma 2.8. *Let ξ be a highest $\ell_{\boldsymbol{s}}$-weight vector. If $1 \leqslant i < j \leqslant \varkappa$ and $1 \leqslant c \leqslant a \leqslant \varkappa$, then we have $t_{ia}(u)t'_{cj}(v)\xi = 0$. Similarly, if $1 \leqslant i \leqslant \varkappa$ and $1 \leqslant c < a \leqslant \varkappa$, then $t_{ia}(u)t'_{ci}(v)\xi = 0$.*

Proof. First we consider the case when $a > c$. Then by (2.10), we have

eq:notsure-001
$$[t_{ia}(u), t'_{cj}(v)]\xi = 0. \tag{2.23}$$

If $c < j$, it is clear from Proposition 2.7 that $t_{ia}(u)t'_{cj}(v)\xi = 0$. If $c \geqslant j$, then $a > c \geqslant j > i$, $t'_{cj}(v)t_{ia}(u)\xi = 0$. It follows from (2.23) that $t_{ia}(u)t'_{cj}(v)\xi = 0$.

Then we consider the case when $a = c$. If $a < j$, then $t_{ia}(u)t'_{aj}(v)\xi = 0$ by Proposition 2.7. If $a \geqslant j$, by (2.10), we have

$$s_a(u-v)[t_{ia}(u), t'_{aj}(v)]\xi = \sum_{c=1}^{\varkappa} t_{ic}(u)t'_{cj}(v)\xi.$$

Note that the right hand side is independent of a. By setting $a = j$ and using Proposition 2.7, we find that

$$\sum_{c=1}^{\varkappa} t_{ic}(u)t'_{cj}(v)\xi = 0.$$

Hence we always have $[t_{ia}(u), t'_{aj}(v)]\xi = 0$. Then again by Proposition 2.7,

$$t_{ia}(u)t'_{aj}(v)\xi = [t_{ia}(u), t'_{aj}(v)]\xi + (-1)^{(|i|+|a|)(|a|+|j|)} t'_{aj}(v)t_{ia}(u)\xi = 0,$$

as $i < j \leqslant a$.

Then we prove the second statement. If $c < i$, then the statement follows from Proposition 2.7. Now suppose that $c \geqslant i$. By (2.10) and the first statement, we have

$$(-1)^{|i||a|+|i||c|+|a||c|}(u-v)[t_{ia}(u), t'_{ci}(v)]\xi = -\sum_{k=1}^{\varkappa} t'_{ck}(v)t_{ka}(u)\xi = 0.$$

Since $a > c \geqslant i$, we have $t'_{ci}(v)t_{ia}(u)\xi = 0$. It follows from the above equation that $t_{ia}(u)t'_{ci}(v)\xi = 0$, completing the proof. □

The following proposition was proved in [RS07, BR09] for the standard parity sequence. The strategy in [RS07, BR09] does not work in general for arbitrary parity sequences.

Let $\rho_k = \sum_{a=k}^{\varkappa} s_a$ for $1 \leqslant k \leqslant \varkappa$. By convention, $\rho_{\varkappa+1} = 0$.

prop:highest-weight-inver

Proposition 2.9. *Let ξ be a highest ℓ-weight vector of highest ℓ-weight $\boldsymbol{\lambda}(u)$. Suppose $\lambda_i'(u)$ is defined as in (2.18), then*

eq:la-prime

$$\lambda_i'(u)=\frac{1}{\lambda_i(u+\rho_{i+1})}\prod_{k=i+1}^{\varkappa}\frac{\lambda_k(u+\rho_k)}{\lambda_k(u+\rho_{k+1})}. \tag{2.24}$$

Proof. For a given parity sequence $\boldsymbol{s}=(s_1,s_2,\cdots,s_\varkappa)\in S_{m|n}$, set $\mathfrak{s}=(s_\varkappa,s_{\varkappa-1},\cdots,s_2,s_1)$. To distinguish generating series for super Yangians of different parity sequences, we shall write $t_{ij}^{\boldsymbol{s}}(u)$, $t_{ij}^{\mathfrak{s}}(u)$, etc. It is also convenient to identify an operator $\sum_{i,j=1}^{\varkappa}(-1)^{|i||j|+|j|}a_{ij}\otimes E_{ij}$ in $\mathscr{Y}_{\boldsymbol{s}}[[u^{-1}]]\otimes \mathrm{End}(V)$ with the matrix $(a_{ij})_{i,j=1}^{\varkappa}$. Then the extra sign ensures that the product of two matrices can still be calculated in the usual way.

Recall the Gauss decomposition of super Yangian $\mathscr{Y}_{\boldsymbol{s}}$, see [Gow07, Pen16]. Let $\mathsf{E}_{ij}^{\boldsymbol{s}}(u)$, $\mathsf{F}_{ji}^{\boldsymbol{s}}(u)$, $\mathsf{D}_{kk}^{\boldsymbol{s}}(u)$, where $1\leqslant i<j\leqslant\varkappa$ and $1\leqslant k\leqslant\varkappa$, be defined by the Gauss decomposition,

$$\begin{aligned}
t_{ii}^{\boldsymbol{s}}(u)&=\mathsf{D}_i^{\boldsymbol{s}}(u)+\sum_{k<i}\mathsf{F}_{ik}^{\boldsymbol{s}}(u)\mathsf{D}_k^{\boldsymbol{s}}(u)\mathsf{E}_{ki}^{\boldsymbol{s}}(u),\\
t_{ij}^{\boldsymbol{s}}(u)&=\mathsf{D}_i^{\boldsymbol{s}}(u)\mathsf{E}_{ij}^{\boldsymbol{s}}(u)+\sum_{k<i}\mathsf{F}_{ik}^{\boldsymbol{s}}(u)\mathsf{D}_k^{\boldsymbol{s}}(u)\mathsf{E}_{kj}^{\boldsymbol{s}}(u),\\
t_{ji}^{\boldsymbol{s}}(u)&=\mathsf{F}_{ji}^{\boldsymbol{s}}(u)\mathsf{D}_i^{\boldsymbol{s}}(u)+\sum_{k<i}\mathsf{F}_{jk}^{\boldsymbol{s}}(u)\mathsf{D}_k^{\boldsymbol{s}}(u)\mathsf{E}_{ki}^{\boldsymbol{s}}(u).
\end{aligned}$$

Similarly, one can define $\mathsf{E}_{ij}^{\mathfrak{s}}(u)$, $\mathsf{F}_{ji}^{\mathfrak{s}}(u)$, $\mathsf{D}_{kk}^{\mathfrak{s}}(u)$.

Let $t_{ij}'^{\boldsymbol{s}}(u)$ correspond to $t_{ij}'(u)$ in $\mathscr{Y}_{\boldsymbol{s}}$. Define similarly $\mathsf{E}_{ij}'^{\boldsymbol{s}}(u)$, $\mathsf{F}_{ji}'^{\boldsymbol{s}}(u)$, $\mathsf{D}_{kk}'^{\boldsymbol{s}}(u)$, for $1\leqslant i<j\leqslant\varkappa$ and $1\leqslant k\leqslant\varkappa$, by

$$\begin{aligned}
t_{ii}'^{\boldsymbol{s}}(u)&=\mathsf{D}_i^{\boldsymbol{s}}(u)^{-1}+\sum_{k>i}\mathsf{E}_{ik}'^{\boldsymbol{s}}(u)\mathsf{D}_k^{\boldsymbol{s}}(u)^{-1}\mathsf{F}_{ki}'^{\boldsymbol{s}}(u),\\
t_{ij}'^{\boldsymbol{s}}(u)&=\mathsf{E}_{ij}'^{\boldsymbol{s}}(u)\mathsf{D}_j^{\boldsymbol{s}}(u)^{-1}+\sum_{k>j}\mathsf{E}_{ik}'^{\boldsymbol{s}}(u)\mathsf{D}_k^{\boldsymbol{s}}(u)^{-1}\mathsf{F}_{kj}'^{\boldsymbol{s}}(u),\\
t_{ji}'^{\boldsymbol{s}}(u)&=\mathsf{D}_j'^{\boldsymbol{s}}(u)^{-1}\mathsf{F}_{ji}'^{\boldsymbol{s}}(u)+\sum_{k>j}\mathsf{E}_{jk}'^{\boldsymbol{s}}(u)\mathsf{D}_k^{\boldsymbol{s}}(u)^{-1}\mathsf{F}_{ki}'^{\boldsymbol{s}}(u).
\end{aligned}$$

Then

eq:app:e-f-inv

$$\begin{aligned}
\mathsf{E}_{ij}'^{\boldsymbol{s}}(u)&=\sum_{i=i_0<i_1<\cdots<i_r=j}(-1)^r\mathsf{E}_{i_0i_1}^{\boldsymbol{s}}(u)\mathsf{E}_{i_1i_2}^{\boldsymbol{s}}(u)\cdots\mathsf{E}_{i_{r-1}i_r}^{\boldsymbol{s}}(u),\\
\mathsf{F}_{ji}'^{\boldsymbol{s}}(u)&=\sum_{i=i_0<i_1<\cdots<i_r=j}(-1)^r\mathsf{F}_{i_ri_{r-1}}^{\boldsymbol{s}}(u)\mathsf{F}_{i_{r-1}i_{r-2}}^{\boldsymbol{s}}(u)\cdots\mathsf{F}_{i_1i_0}^{\boldsymbol{s}}(u).
\end{aligned}\tag{2.25}$$

There exists an isomorphism between $\mathscr{Y}_{\mathfrak{s}}$ and $\mathscr{Y}_{\boldsymbol{s}}$ given by the map

eq:identifi

$$t_{\varkappa+1-j,\varkappa+1-i}^{\mathfrak{s}}(u)\to(-1)^{|i||j|+|j|}t_{ij}'^{\boldsymbol{s}}(u),\quad 1\leqslant i,j\leqslant\varkappa, \tag{2.26}$$

where the signs $|i|$ and $|j|$ are determined by the parity sequence $\boldsymbol{s}$.

We shall identify $t_{ij}^{\mathfrak{s}}(u)$ with $t_{\varkappa+1-j,\varkappa+1-i}'^{\boldsymbol{s}}(u)$ with certain signs as in (2.26). With this identification, when $\varkappa=3$, one has

$$T^{\mathfrak{s}}(u)=\begin{pmatrix}
\mathsf{D}_3^{\boldsymbol{s}} & \mathsf{E}_{23}^{\boldsymbol{s}}(\mathsf{D}_3^{\boldsymbol{s}})^{-1} & \mathsf{E}_{13}^{\boldsymbol{s}}(\mathsf{D}_3^{\boldsymbol{s}})^{-1}\\
(\mathsf{D}_3^{\boldsymbol{s}})^{-1}\mathsf{F}_{32}^{\boldsymbol{s}} & (\mathsf{D}_2^{\boldsymbol{s}})^{-1}+\mathsf{E}_{23}^{\boldsymbol{s}}(\mathsf{D}_3^{\boldsymbol{s}})^{-1}\mathsf{F}_{32}^{\boldsymbol{s}} & \mathsf{E}_{12}^{\boldsymbol{s}}(\mathsf{D}_2^{\boldsymbol{s}})^{-1}+\mathsf{E}_{13}^{\boldsymbol{s}}(\mathsf{D}_3^{\boldsymbol{s}})^{-1}\mathsf{F}_{32}^{\boldsymbol{s}}\\
(\mathsf{D}_3^{\boldsymbol{s}})^{-1}\mathsf{F}_{31}^{\boldsymbol{s}} & (\mathsf{D}_2^{\boldsymbol{s}})^{-1}\mathsf{F}_{21}^{\boldsymbol{s}}+\mathsf{E}_{23}^{\boldsymbol{s}}(\mathsf{D}_3^{\boldsymbol{s}})^{-1}\mathsf{F}_{31}^{\boldsymbol{s}} & (\mathsf{D}_1^{\boldsymbol{s}})^{-1}+\mathsf{E}_{12}^{\boldsymbol{s}}(\mathsf{D}_2^{\boldsymbol{s}})^{-1}\mathsf{F}_{21}^{\boldsymbol{s}}+\mathsf{E}_{13}^{\boldsymbol{s}}(\mathsf{D}_3^{\boldsymbol{s}})^{-1}\mathsf{F}_{31}^{\boldsymbol{s}}
\end{pmatrix},$$

cf. [LPRS19, equaltion (B.4)]. Here we drop the spectral parameter u and the signs for brevity.

Under the identification above, one proves similarly to [LPRS19, Theorem 4.2] for the super Yangian $\mathscr{Y}_{\boldsymbol{s}}$ that

eq:cartan-current-new

$$\begin{aligned}
\mathsf{E}^{\mathfrak{s}}_{\varkappa+1-j,\varkappa+1-i}(u) &= (-1)^{|i||j|+|j|}\mathsf{E}'^{\boldsymbol{s}}_{ij}(u+\rho_j), && 1\leqslant i<j\leqslant \varkappa,\\
\mathsf{F}^{\mathfrak{s}}_{\varkappa+1-i,\varkappa+1-j}(u) &= (-1)^{|i||j|+|i|}\mathsf{F}'^{\boldsymbol{s}}_{ji}(u+\rho_j), && 1\leqslant i<j\leqslant \varkappa,\\
\mathsf{D}^{\mathfrak{s}}_{\varkappa+1-k}(u) &= \frac{1}{\mathsf{D}^{\boldsymbol{s}}_k(u+\rho_{k+1})}\prod_{a=k+1}^{\varkappa}\frac{\mathsf{D}^{\boldsymbol{s}}_a(u+\rho_a)}{\mathsf{D}^{\boldsymbol{s}}_a(u+\rho_{a+1})}, && 1\leqslant k\leqslant \varkappa.
\end{aligned}\tag{2.27}$$

Now we are ready to prove Proposition 2.9.

It is well known that for a highest $\ell_{\boldsymbol{s}}$-weight vector v of highest weight $\boldsymbol{\lambda}(u)$, we have

eq:app:1

$$\mathsf{E}^{\boldsymbol{s}}_{ij}(u)v=0,\qquad t^{\boldsymbol{s}}_{ii}(u)v=\mathsf{D}^{\boldsymbol{s}}_i(u)v=\lambda_i(u)v,\tag{2.28}$$

see e.g. [Lu22, Section 2.5]. By Gauss decomposition,

$$t^{\mathfrak{s}}_{ii}(u)=\mathsf{D}^{\mathfrak{s}}_i(u)+\sum_{k<i}\mathsf{F}^{\mathfrak{s}}_{ik}(u)\mathsf{D}^{\mathfrak{s}}_k(u)\mathsf{E}^{\mathfrak{s}}_{ki}(u),$$

it follows from (2.25) and (2.28) that $t^{\mathfrak{s}}_{ii}(u)v=\mathsf{D}^{\mathfrak{s}}_i(u)v$. Therefore, by (2.27) and (2.28) that

$$t'^{\boldsymbol{s}}_{ii}(u)v=t^{\mathfrak{s}}_{\varkappa+1-i,\varkappa+1-i}(u)v=\mathsf{D}^{\mathfrak{s}}_{\varkappa+1-i}(u)v=\frac{1}{\lambda_i(u+\rho_{i+1})}\prod_{k=i+1}^{\varkappa}\frac{\lambda_k(u+\rho_k)}{\lambda_k(u+\rho_{k+1})}v,$$

completing the proof of Proposition 2.9. ☐

3. Twisted super Yangian of type AIII

sec:twisted-super-yangians

3.1. **Definition.** Fix a sequence of integers $\boldsymbol{\varepsilon}=(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_\varkappa)$, where $\varepsilon_i\in\{\pm1\}$. Denote by $G^{\boldsymbol{\varepsilon}}$ the diagonal $\varkappa\times\varkappa$ (super)matrix

eq:G

$$G^{\boldsymbol{\varepsilon}}=\mathrm{diag}(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_\varkappa).\tag{3.1}$$

The matrix $G^{\boldsymbol{\varepsilon}}$ satisfies the reflection equation

eq:reflectG

$$R(u-v)G^{\boldsymbol{\varepsilon}}_1R(u+v)G^{\boldsymbol{\varepsilon}}_2=G^{\boldsymbol{\varepsilon}}_2R(u+v)G^{\boldsymbol{\varepsilon}}_1R(u-v).\tag{3.2}$$

Definition 3.1 ([MR02, RS07, BR09])**.** *The twisted super Yangian of type AIII $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$ is a $\mathbb{Z}_2$-graded unital associative algebra over $\mathbb{C}$ with generators $\{b^{(r)}_{ij}\mid 1\leqslant i,j\leqslant\varkappa,\ r\geqslant1\}$ and defining relations given by*

eq:comm-series b

$$\begin{aligned}
[b_{ij}(u),b_{kl}(v)] &= \frac{(-1)^{|i||j|+|i||k|+|j||k|}}{u-v}\big(b_{kj}(u)b_{il}(v)-b_{kj}(v)b_{il}(u)\big)\\
&\quad+\frac{(-1)^{|i||j|+|i||k|+|j||k|}}{u+v}\Big(\delta_{kj}\sum_{a=1}^{\varkappa}b_{ia}(u)b_{al}(v)-\delta_{il}\sum_{a=1}^{\varkappa}b_{ka}(v)b_{aj}(u)\Big)\\
&\quad-\frac{1}{u^2-v^2}\delta_{ij}\Big(\sum_{a=1}^{\varkappa}b_{ka}(u)b_{al}(v)-\sum_{a=1}^{\varkappa}b_{ka}(v)b_{al}(u)\Big)
\end{aligned}\tag{3.3}$$

and the unitary condition

eq:unitary-series

$$\sum_{a=1}^{\varkappa}b_{ia}(u)b_{aj}(-u)=\delta_{ij},\tag{3.4}$$

where

$$b_{ij}(u)=\sum_{k=0}^{\infty}b^{(k)}_{ij}u^{-k},\qquad b^{(0)}_{ij}=\delta_{ij}\varepsilon_i,$$

and the generators $b^{(r)}_{ij}$ have the parity $|i|+|j|$.

Define the operator $B(u) \in B_{\boldsymbol{s},\boldsymbol{\varepsilon}}[[u^{-1}]] \otimes \mathrm{End}(V)$,

$$B(u) = \sum_{i,j=1}^{\varkappa} (-1)^{|i||j|+|j|} b_{ij}(u) \otimes E_{ij}. \qquad \text{eq matrix notation b} \tag{3.5}$$

Then the defining relations of $B_{\boldsymbol{s},\boldsymbol{\varepsilon}}$ can also be written as the reflection equation

$$R(u-v)B_1(u)R(u+v)B_2(v) = B_2(v)R(u+v)B_1(u)R(u-v) \qquad \text{eq:comm-generators b} \tag{3.6}$$

and

$$B(u)B(-u) = 1. \qquad \text{eq:unitary} \tag{3.7}$$

We shall also use the algebra $\widetilde{\mathcal{B}}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$ defined in the same way as $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$ but with the unitary condition (3.4) omitted. Since there are no other types in this paper, we shall simply call $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$ and $\widetilde{\mathcal{B}}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$ *twisted super Yangian* and *extended twisted super Yangian*, respectively.

The extended twisted super Yangian (reflection superalgebra) previously appeared in [RS07, BR09] on the study of Bethe ansatz for open spin chains with diagonal boundary conditions. Certain properties on $\widetilde{\mathcal{B}}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$ has been obtained in [RS07, BR09]. We shall reproduce some of them.

Proposition 3.2. *In the extended twisted super Yangian $\widetilde{\mathcal{B}}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$, the product $B(u)B(-u)$ is a scalar matrix*

$$B(u)B(-u) = f(u)1, \qquad \text{eq:bb-central} \tag{3.8}$$

where $f(u)$ is a series in u^{-2} whose coefficients are central in $\widetilde{\mathcal{B}}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$.

Proof. The proof is parallel to that of [MR02, Proposition 2.1]. Multiplying both sides of (3.3) by $u^2 - v^2$ and set $v = -u$, one has

$$\begin{aligned}(-1)^{|i||j|+|i||k|+|j||k|}2u\Big(\delta_{kj} \sum_{a=1}^{\varkappa} b_{ia}(u)b_{al}(v) - \delta_{il}\sum_{a=1}^{\varkappa} b_{ka}(v)b_{aj}(u)\Big)& \\ = \delta_{ij}\Big(\sum_{a=1}^{\varkappa} b_{ka}(u)b_{al}(v) - \sum_{a=1}^{\varkappa} b_{ka}(v)b_{al}(u)\Big)&.\end{aligned} \tag{3.9}$$

By taking suitable indices i, j, k, l, one obtains that

$$B(u)B(-u) = B(-u)B(u)$$

and the matrix is indeed a scalar matrix. Therefore, (3.8) holds and in particular $f(u)$ is a series in u^{-2} as $B(u)B(-u)$ is even.

Multiplying both side of (3.6) by $B_2(-v)$ from the right, we have

$$\begin{aligned} R(u-v)B_1(u)R(u+v)f(v) &= B_2(v)R(u+v)B_1(u)R(u-v)B_2(-v) \\ &\overset{(3.6)}{=} B_2(v)B_2(-v)R(u-v)B_1(u)R(u+v) \\ &= f(v)R(u-v)B_1(u)R(u+v). \end{aligned}$$

Therefore, the coefficients of $f(v)$ are central in $\widetilde{\mathcal{B}}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$. □

Let $\hbar(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$ be such that $\hbar(u)\hbar(-u) = 1$. There is an automorphism $\mathscr{M}_{\hbar(u)}$ defined by

$$\mathscr{M}_{\hbar(u)} : \mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}} \to \mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}, \quad B(u) \mapsto \hbar(u)B(u). \qquad \text{mhu} \tag{3.10}$$

sec:properties

3.2. **Basic properties of twisted super Yangian.** In this section, we collect some basic properties of the twisted super Yangian $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$.

thm:embedding

Proposition 3.3. *The mapping*

eq:emd-b

$$\varphi : B(u) \to T(u)G^{\varepsilon}T^{-1}(-u) \tag{3.11}$$

defines an embedding which identify the twisted Yangian $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$ as a subalgebra of the super Yangian $\mathcal{Y}_{\boldsymbol{s}}$.

Proof. The proof is essentially the same as that of [MR02, Theorem 3.1]. We first check that φ induces a superalgebra homomorphism which we again denote by φ and then prove that this superalgebra homomorphism φ is injective.

For brevity, we simply write G for G^{ε}.

Set $S(u) = T(u)GT^{-1}(-u)$, then we immediately have

$$S(u)S(-u) = T(u)GT^{-1}(-u)T(-u)GT^{-1}(u) = 1$$

which verifies the unitary condition (3.7).

On the other hand, we also have

$$\begin{aligned} R(u-v)S_1(u)R(u+v)S_2(v) &= R(u-v)T_1(u)G_1T_1^{-1}(-u)R(u+v)T_2(v)G_2T_2^{-1}(-v) \\ &\overset{(2.9)}{=} R(u-v)T_1(u)G_1T_2(v)R(u+v)T_1^{-1}(-u)G_2T_2^{-1}(-v) \\ &= R(u-v)T_1(u)T_2(v)G_1R(u+v)G_2T_1^{-1}(-u)T_2^{-1}(-v) \\ &\overset{(2.4)}{=} T_2(v)T_1(u)R(u-v)G_1R(u+v)G_2T_1^{-1}(-u)T_2^{-1}(-v) \\ &\overset{(3.2)}{=} T_2(v)T_1(u)G_2R(u+v)G_1R(u-v)T_1^{-1}(-u)T_2^{-1}(-v) \\ &\overset{(2.4)}{=} T_2(v)T_1(u)G_2R(u+v)G_1T_2^{-1}(-v)T_1^{-1}(-u)R(u-v) \\ &= T_2(v)G_2T_1(u)R(u+v)T_2^{-1}(-v)G_1T_1^{-1}(-u)R(u-v) \\ &\overset{(2.9)}{=} T_2(v)G_2T_2^{-1}(-v)R(u+v)T_1(u)G_1T_1^{-1}(-u)R(u-v) \\ &= S_2(v)R(u+v)S_1(u)R(u-v). \end{aligned}$$

Therefore, $S(u)$ also satisfies the reflection equation (3.6).

Then we show that φ is injective. Introduce the filtration on $\mathcal{Y}_{\boldsymbol{s}}$ defined by $\deg_1 t_{ij}^{(r)} = r$, see [Gow07], and a similar filtration on $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$ by setting $\deg_1 b_{ij}^{(r)} = r$. Note that for the matrix elements of $S(u)$, we have

eq:embed-expl

$$\mathtt{s}_{ij}(u) = \varepsilon_i\delta_{ij} + \sum_{r>0} \mathtt{s}_{ij}^{(r)}u^{-r} = \sum_{a=1}^{\varkappa} \varepsilon_a t_{ia}(u)t'_{aj}(-u). \tag{3.12}$$

It follows from (2.8) that the degree of $\mathtt{s}_{ij}^{(r)}$ is at most r. Therefore φ preserves the filtration and hence induces a homomorphism of the associated graded superalgebras

$$\overline{\varphi} : \mathrm{gr}_1\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}} \to \mathrm{gr}_1\mathcal{Y}_{\boldsymbol{s}}.$$

Denote by $\bar{t}_{ij}^{(r)}$ the image of $t_{ij}^{(r)}$ in the r-th component of $\mathrm{gr}_1\mathcal{Y}_{\boldsymbol{s}}$.

It is clear from (2.2) that $\mathrm{gr}_1\mathcal{Y}_{\boldsymbol{s}}$ is supercommutative, and moreover it follows from [Gow07, Theorem 1] that these elements $\bar{t}_{ij}^{(r)}$ are algebraically independent generators (in the super sense). It is also clear that $\mathrm{gr}_1\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$ is supercommutative. Denote by $\bar{b}_{ij}^{(r)}$ the image of $b_{ij}^{(r)}$ in the r-th component of $\mathrm{gr}_1\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$. Due to the unitary condition (3.4), the elements

eq:ind-generator-b

$$\begin{aligned} &\bar{b}_{ij}^{(2p-1)}, &\quad &\text{if } \varepsilon_i=\varepsilon_j,\\ &\bar{b}_{ij}^{(2p)}, &\quad &\text{if } \varepsilon_i\neq\varepsilon_j, \end{aligned} \tag{3.13}$$

for $1\leqslant i,j\leqslant \varkappa$ and $p\in\mathbb{Z}_{>0}$, generate the superalgebra $\mathrm{gr}_1\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$. By (3.12) and (2.8), we find that

eq:graded-generators

$$\overline{\varphi}:\bar{b}_{ij}^{(r)}\mapsto ((-1)^{r-1}\varepsilon_i+\varepsilon_j)\bar{t}_{ij}^{(r)}+\cdots. \tag{3.14}$$

Here $\cdots$ stands for a linear combination of monomials in $\bar{t}_{ab}^{(p)}$ with $p<r$ for various $1\leqslant a,b\leqslant\varkappa$. therefore, the elements in (3.13) are algebraically independent, completing the proof. $\square$

We immediately have the following PBW-type theorem for the twisted super Yangian $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$.

cor:PBW

Corollary 3.4. *Given any total ordering on the elements*

$$\begin{aligned} &b_{ij}^{(2p-1)}, &\quad &\textit{if } \varepsilon_i=\varepsilon_j,\\ &b_{ij}^{(2p)}, &\quad &\textit{if } \varepsilon_i\neq\varepsilon_j, \end{aligned} \tag{3.15}$$

for $1\leqslant i,j\leqslant\varkappa$ and $p\in\mathbb{Z}_{>0}$, the ordered monomials in these elements, containing no second or higher order powers of the odd generators, form a basis of the twisted super Yangian $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$.

Thanks to Proposition 3.3, the twisted super Yangian is identified with a subalgebra of $\mathcal{Y}_{\boldsymbol{s}}$ by identifying $b_{ij}(u)$ with $\mathsf{s}_{ij}(u)$. As the twisted super Yangians of types AI and AII, $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$ is also a coideal subalgebra of $\mathcal{Y}_{\boldsymbol{s}}$.

prop:coproduct

Proposition 3.5. *The subalgebra $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$ is a left coideal subalgebra in $\mathcal{Y}_{\boldsymbol{s}}$,*

eq:b-copro-in-t

$$\Delta(b_{ij}(u))=\sum_{a,c=1}^{\varkappa} t_{ia}(u)t'_{cj}(-u)\otimes b_{ac}(u)(-1)^{(|c|+|j|)(|a|+|c|)}. \tag{3.16}$$

Proof. Note that Δ is a superalgebra homomorphism. One finds

$$\Delta(t'_{ij}(u))=\sum_{a=1}^{\varkappa} t'_{aj}(u)\otimes t'_{ia}(u)(-1)^{(|a|+|j|)(|i|+|a|)}.$$

Then the statement follows from a straightforward computation. $\square$

Let θ be the involution of $\mathfrak{gl}_{m|n}^{\boldsymbol{s}}$ sending e_{ij} to $\varepsilon_i\varepsilon_j e_{ij}$ and $(\mathfrak{gl}_{m|n}^{\boldsymbol{s}})^\theta$ the fixed point Lie subalgebra of $\mathfrak{gl}_{m|n}^{\boldsymbol{s}}$ under θ. Note that θ depends on the diagonal matrix $G^{\boldsymbol{\varepsilon}}$ which we shall not write explicitly. Then $(\mathfrak{gl}_{m|n}^{\boldsymbol{s}},(\mathfrak{gl}_{m|n}^{\boldsymbol{s}})^\theta)$ is a (super)symmetric pair of type AIII, cf. [She22]. Write $\mathfrak{gl}_{m|n}^{\boldsymbol{s}}=\mathfrak{k}+\mathfrak{p}$ as the (±1)-eigenspace decomposition with respect to θ. In particular,

$$\mathfrak{k}=(\mathfrak{gl}_{m|n}^{\boldsymbol{s}})^\theta\cong\mathfrak{gl}_{m_1|n_1}\oplus\mathfrak{gl}_{m_2|n_2},$$

where

$$\begin{aligned} m_1&=\#\{i\,|\,s_i=\varepsilon_i=1,\ 1\leqslant i\leqslant\varkappa\}, &\qquad n_1&=\#\{i\,|\,-s_i=\varepsilon_i=1,\ 1\leqslant i\leqslant\varkappa\},\\ m_2&=\#\{i\,|\,s_i=-\varepsilon_i=1,\ 1\leqslant i\leqslant\varkappa\}, &\qquad n_2&=\#\{i\,|\,s_i=\varepsilon_i=-1,\ 1\leqslant i\leqslant\varkappa\}. \end{aligned}$$

Clearly, a basis of $\mathfrak{k}$ is given by all e_{ij} for $1\leqslant i,j\leqslant\varkappa$ and $\varepsilon_i=\varepsilon_j$, while a basis of $\mathfrak{k}$ is given by all e_{ij} for $1\leqslant i,j\leqslant\varkappa$ and $\varepsilon_i\neq\varepsilon_j$.

Extend the involution θ on $\mathfrak{gl}_{m|n}^{\boldsymbol{s}}$ to $\hat{\theta}$ on $\mathfrak{gl}_{m|n}^{\boldsymbol{s}}[x]$ by sending

$$\hat{\theta}(gx^k)=\theta(g)(-x)^k,$$

for $g \in \mathfrak{gl}^{\boldsymbol{s}}_{m|n}$ and $k \in \mathbb{Z}_{\geqslant 0}$. Let $\mathfrak{gl}^{\boldsymbol{s}}_{m|n}[x]^{\hat{\theta}}$ be the fixed point subalgebra of $\mathfrak{gl}^{\boldsymbol{s}}_{m|n}[x]$ under $\hat{\theta}$. Then we have

$$\mathfrak{gl}^{\boldsymbol{s}}_{m|n}[x]^{\hat{\theta}} = (\mathfrak{k} \otimes \mathbb{C}[x^2]) \bigoplus (\mathfrak{p} \otimes x\mathbb{C}[x^2]).$$

There is also another filtration of $\mathcal{Y}_{\boldsymbol{s}}$ defined by setting $\deg_2 t_{ij}^{(r)} = r-1$. It is well known [Gow07] that the associated graded superalgebra $\mathrm{gr}_2\mathcal{Y}_{\boldsymbol{s}}$ is the universal enveloping superalgebra $\mathrm{U}(\mathfrak{gl}_{m|n}[x])$ and the correspondence is given by

eq:griso
$$\mathrm{U}(\mathfrak{gl}_{m|n}[x]) \to \mathrm{gr}_2\, \mathcal{Y}_{\boldsymbol{s}}, \qquad e_{ij}x^r \mapsto s_i \bar{t}_{ij}^{(r+1)}. \tag{3.17}$$

Regard $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$ as a subalgebra of $\mathcal{Y}_{\boldsymbol{s}}$ via Proposition 3.3. Then we have the filtration on $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$ given by $\deg_2 b_{ij}^{(r)} = r-1$. Let $\mathrm{gr}_2\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$ be the associated graded superalgebra.

prop:limit

Proposition 3.6. *The twisted super Yangian $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$ is a deformation of $\mathrm{U}(\mathfrak{gl}^{\boldsymbol{s}}_{m|n}[x]^{\hat{\theta}})$,*

$$\mathrm{gr}_2\, \mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}} \cong \mathrm{U}(\mathfrak{gl}^{\boldsymbol{s}}_{m|n}[x]^{\hat{\theta}}).$$

Proof. For $r \in \mathbb{Z}_{\geqslant 0}$, let $\bar{b}_{ij}^{(r+1)}$ be the image of $b_{ij}^{(r+1)}$ in the r-th component of $\mathrm{gr}_2\,\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$. It follows from the proof of Proposition 3.3 that under the isomorphism (3.17),

$$s_i((-1)^{r-1}\varepsilon_i + \varepsilon_j)e_{ij}x^{r-1} \mapsto ((-1)^{r-1}\varepsilon_i + \varepsilon_j)\bar{t}_{ij}^{(r)} = \bar{b}_{ij}^{(r)}.$$

Note that $\mathrm{U}(\mathfrak{gl}^{\boldsymbol{s}}_{m|n}[x]^{\hat{\theta}})$ (resp. $\mathrm{gr}_2\,\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$) is generated by $((-1)^{r-1}\varepsilon_i+\varepsilon_j)e_{ij}x^{r-1}$ (resp. $\bar{b}_{ij}^{(r)}$), for $1 \leqslant i,j \leqslant \varkappa$ and $r \in \mathbb{Z}_{>0}$, the proposition follows. ☐

4. Highest weight representations

sec:reps

In this section, we discuss the highest representations of the twisted super Yangian $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$.

4.1. Highest weight representations.

Similar to [MR02], we define the highest weight representation of $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$ as follows.

Definition 4.1. *A representation V of $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$ is called highest $\ell_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-weight if there exists a nonzero vector $\eta \in V$ such that V is generated by η and η satisfies*

eq:highest-B
$$\begin{aligned} &b_{ij}(u)\eta = 0, && 1 \leqslant i < j \leqslant \varkappa,\\ &b_{ii}(u)\eta = \mu_i(u)\eta, && 1 \leqslant i \leqslant \varkappa, \end{aligned} \tag{4.1}$$

where $\mu_i(u) \in \varepsilon_i + u^{-1}\mathbb{C}[[u^{-1}]]$. The vector η is called a highest $\ell_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-weight vector of V and the tuple $\boldsymbol{\mu}(u) = (\mu_i(u))_{1\leqslant i\leqslant\varkappa}$ is the highest $\ell_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-weight of V.

eg:1-dim

Example 4.2. For any $\gamma \in \mathbb{C}$, there exists a one-dimensional module $\mathbb{C}_\gamma := \mathbb{C}\eta_\gamma$ generated by a highest $\ell_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-weight vector η_γ such that

$$b_{ij}(u)\eta_\gamma = \delta_{ij}\frac{\varepsilon_i u + \gamma}{u-\gamma}\eta_\gamma.$$ ☐

We have the following standard statements.

By the relations (3.3), we have

$$[b_{ij}^{(1)}, b_{kl}(u)] = (-1)^{|i||j|+|i||k|+|j||k|}(\varepsilon_i+\varepsilon_j)(\delta_{kj}b_{il}(u) - \delta_{il}b_{kj}(u)).$$

In particular, we have

eq:bii-weightb
$$\big[s_i\varepsilon_i b_{ii}^{(1)}/2, b_{kl}(u)\big] = \delta_{ki}b_{il}(u) - \delta_{il}b_{kj}(u). \tag{4.2}$$

Therefore, the operators $s_i\varepsilon_i b_{ii}^{(1)}/2$ are pairwise commuting.

We say that a $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-module V has a $\mathfrak{gl}^{\boldsymbol{s}}_{m|n}$-weight subspace decomposition if it possesses a common eigenspace decomposition for the commuting operators $b_{ii}^{(1)}$, $1 \leqslant i \leqslant \varkappa$. We say that a vector $v \in V$ has weight $w = (w_1, \ldots, w_\varkappa)$ if

$$\frac{1}{2} s_i \varepsilon_i b_{ii}^{(1)} v = w_i v, \quad 1 \leqslant i \leqslant \varkappa.$$

Denote by $(V)_w$ the weight subspace of V with weight w.

Note that under the identification (3.12), we have

$$b_{ii}^{(1)} = 2\varepsilon_i t_{ii}^{(1)} = 2 s_i \varepsilon_i e_{ii}.$$

Therefore, the definition is compatible with (2.15) if we consider a $\mathcal{Y}_{\boldsymbol{s}}$-module as a $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-module by restriction. For a highest $\ell_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-weight $\boldsymbol{\mu}(u)$, we define a $\mathfrak{gl}^{\boldsymbol{s}}_{m|n}$-weight $\varpi(\boldsymbol{\mu}(u))$, similar to (2.15), associated to it by the rule

varpi2
$$\varpi(\boldsymbol{\mu}(u))(e_{ii}) = \frac{1}{2} s_i \varepsilon_i \mu_{i,1}, \tag{4.3}$$

where $\mu_{i,1}$ is the coefficient of u^{-1} in the series $\mu_i(u)$. Then a highest $\ell_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-weight vector of $\mathfrak{gl}^{\boldsymbol{s}}_{m|n}$-weight $\varpi(\boldsymbol{\mu}(u))$.

lem:wt-changeb

Lemma 4.3. *We have*

$$b_{ij}^{(r)} (V)_w \subset (V)_{w+\epsilon_i-\epsilon_j},$$

for $1 \leqslant i, j \leqslant \varkappa$, *and* $r \in \mathbb{Z}_{>0}$.

Proof. The lemma follows from (4.2) by a direct computation. □

Thus, we have the following corollary of Corollary 3.4 and Lemma 4.3.

cor:wt-changeb

Corollary 4.4. *If V is a $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-module of highest $\ell_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-weight $\boldsymbol{\mu}(u)$, then V has a $\mathfrak{gl}^{\boldsymbol{s}}_{m|n}$-weight subspace decomposition. Moreover, its highest $\ell_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-weight vector has weight $\varpi(\boldsymbol{\mu}(u))$ and the other weight vectors have weights that are strictly smaller (with respect to $\geqslant_{\boldsymbol{s}}$ defined in §2.1) than $\varpi(\boldsymbol{\mu}(u))$.* □

Let V be a representation of $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$. Set

$$V^\circ = \{\eta \in V \mid b_{ij}(u)\eta = 0,\ 1 \leqslant i < j \leqslant \varkappa\}.$$

lem:nontrivial

Lemma 4.5. *If V is a finite-dimensional representation of $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$, then V° is nontrivial.*

Proof. Since the operators $s_i\varepsilon_i b_{ii}^{(1)}/2$ are pairwise commuting and hence have at least a common eigenvector $\tilde{\eta} \neq 0$ in V. Suppose $V^\circ = 0$, then there exists an infinite sequence of nonzero vectors in V,

$$\tilde{\eta}, \quad b_{i_1 j_1}^{(r_1)} \tilde{\eta}, \quad b_{i_2 j_2}^{(r_2)} b_{i_1 j_1}^{(r_1)} \tilde{\eta}, \quad \cdots,$$

where $i_k < j_k$ and $r_k > 0$ for all $k \in \mathbb{Z}_{>0}$. It follows from Lemma 4.3 and Corollary 4.4 that the above vectors have different $\mathfrak{gl}^{\boldsymbol{s}}_{m|n}$-weights. Therefore, they must be linearly independent and hence we obtain a contradiction as V is finite-dimensional, completing the proof. □

Throughout the paper, for $X, X' \in \mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$, we shall write $X \equiv X'$ if $X - X'$ belongs to the left ideal of $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$ generated by the coefficients of $b_{ij}(u)$ for $1 \leqslant i < j \leqslant \varkappa$.

lem:invariant

Lemma 4.6. *The space V° is invariant under the operators $b_{rr}(u)$, for $1 \leqslant r \leqslant \varkappa$.*

Proof. We prove $b_{ij}(u)b_{rr}(v) \equiv 0$ for $1 \leqslant i < j \leqslant \varkappa$ and $1 \leqslant r \leqslant \varkappa$ by a reverse induction on r.

For the base case $r = \varkappa$, it is immediate from (3.3) that $b_{ij}(u)b_{\varkappa\varkappa}(v) \equiv 0$ for $i < j < \varkappa$. Similarly, for $i < \varkappa$, we obtain

$$b_{i\varkappa}(u)b_{\varkappa\varkappa}(v) \equiv \frac{(-1)^{|i||j|+|i||k|+|j||k|}}{u+v} b_{i\varkappa}(u)b_{\varkappa\varkappa}(v),$$

which implies $b_{i\varkappa}(u)b_{\varkappa\varkappa}(u) \equiv 0$. Therefore, the base case is established.

Now let $r < \varkappa$. For $i < j$ and $i < k$, it follows from (3.3) that

lem:commun-1
$$s_j b_{ij}(u) b_{jk}(v) \equiv \frac{1}{u+v} \sum_{a=k}^{\varkappa} b_{ia}(u) b_{ak}(v). \tag{4.4}$$

Note that the right hand side of (4.4) is independent of j. Therefore, for j and j' such that $i < j, j'$, we have

lem:commun-2
$$s_j b_{ij}(u) b_{jk}(v) \equiv s_{j'} b_{ij'}(u) b_{j'k}(v). \tag{4.5}$$

In the following, we always assume that $i < j$. We have four cases.

(1) The case $i < r$ and $j \neq r$. It is straightforward from (3.3) that $b_{ij}(u) b_{rr}(v) \equiv 0$.

(2) The case $i < r$ and $j = r$. By (4.4) and (4.5), we also have

$$s_r b_{ir}(u) b_{rr}(v) \equiv \frac{s_r}{u+v} b_{ir}(u) b_{rr}(v) \sum_{a=r}^{\varkappa} s_a,$$

which gives $b_{ir}(u) b_{rr}(v) \equiv 0$.

(3) The case $r < i < j$. Using (3.3) for $[b_{rr}(v), b_{ij}(u)]$, we have

$$\begin{aligned} b_{ij}(u) b_{rr}(v) &\equiv \frac{1}{u^2 - v^2} \Big(\sum_{a=j}^{\varkappa} b_{ia}(u) b_{aj}(v) - \sum_{a=j}^{\varkappa} b_{ia}(v) b_{aj}(u) \Big) \\ &\overset{(4.5)}{\equiv} \frac{s_j}{u^2 - v^2} \big(b_{ij}(u) b_{jj}(v) - b_{ij}(v) b_{jj}(u) \big) \sum_{a=j}^{\varkappa} s_a. \end{aligned}$$

Thus, $b_{ij}(u) b_{rr}(v) \equiv 0$ as $b_{ij}(u) b_{jj}(v) \equiv b_{ij}(v) b_{jj}(u) \equiv 0$ by induction hypothesis.

(4) The case $r = i < j$. By (3.3), we have

$$s_r b_{rj}(u) b_{rr}(v) \equiv \frac{1}{u-v} \big(b_{rj}(u) b_{rr}(v) - b_{rj}(v) b_{rr}(u) \big) - \frac{1}{u+v} \sum_{a=j}^{\varkappa} b_{ra}(v) b_{aj}(u).$$

Note that by (4.5), $s_a b_{ra}(v) b_{aj}(u) \equiv s_j b_{rj}(v) b_{jj}(u) \equiv 0$ for $j \leqslant a \leqslant \varkappa$ by induction hypothesis. We obtain that

lem:commun-3
$$\frac{u - v - s_r}{u - v} b_{rj}(u) b_{rr}(v) + \frac{s_r}{u-v} b_{rj}(v) b_{rr}(u) \equiv 0. \tag{4.6}$$

Interchanging u and v, we also have

lem:commun-4
$$-\frac{s_r}{u-v} b_{rj}(u) b_{rr}(v) + \frac{u - v + s_r}{u - v} b_{rj}(v) b_{rr}(u) \equiv 0. \tag{4.7}$$

The system of equations (4.6) and (4.7) has only zero solution, therefore we conclude that

$$b_{rj}(u) b_{rr}(v) \equiv 0.$$

The proof now is complete. $\square$

lem:commute

Lemma 4.7. *All the operators $b_{rr}(u)$, $1 \leqslant r \leqslant \varkappa$, on V° commute.*

Proof. For any $1 \leqslant r \leqslant \varkappa$, it follows from (3.3) that

lem:commun-5
$$\Big(1 - \frac{s_r}{u+v} \Big) [b_{rr}(u), b_{rr}(v)] \equiv \frac{s_r}{u+v} \iota_r(u,v), \tag{4.8}$$

where

lem:commun-6
$$\iota_r(u,v) = \sum_{a=r+1}^{\varkappa} \big(b_{ra}(u) b_{ar}(v) - b_{ra}(v) b_{ar}(u) \big). \tag{4.9}$$

Again by (3.3), for $a > r$, we have

$$b_{ra}(u) b_{ar}(v) \equiv \frac{s_a}{u-v} \big(b_{aa}(u) b_{rr}(v) - b_{aa}(v) b_{rr}(u) \big) + \frac{s_a}{u+v} \Big(\sum_{c=r}^{\varkappa} b_{rc}(u) b_{cr}(v) - \sum_{c=a}^{\varkappa} b_{ac}(v) b_{ca}(u) \Big).$$

Switching u and v and taking the difference, we obtain

$$\begin{aligned} b_{ra}(u)b_{ar}(v)- & \ b_{ra}(v)b_{ar}(u) \\ & \equiv \frac{s_a}{u+v}\Big([b_{rr}(u),b_{rr}(v)]+[b_{aa}(u),b_{aa}(v)]+\iota_r(u,v)+\iota_a(u,v)\Big). \end{aligned}$$

Summing over a, we obtain

lem:commun-7

$$(u+v-\rho_{r+1})\iota_r(u,v)\equiv \rho_{r+1}[b_{rr}(u),b_{rr}(v)]+\sum_{a=r+1}^{\varkappa} s_a\Big([b_{aa}(u),b_{aa}(v)]+\iota_a(u,v)\Big) \tag{4.10}$$

Using (4.8) and (4.10), one easily shows that $[b_{rr}(u),b_{rr}(v)]\equiv 0$ and $\iota_r(u,v)\equiv 0$ by a reverse induction on r. Therefore, if $i<r$, it follows from (3.3) that

$$[b_{ii}(u),b_{rr}(v)]\equiv -\frac{1}{u^2-v^2}\Big([b_{rr}(u),b_{rr}(v)]+\iota_r(u,v)\Big)\equiv 0.$$

Hence we proved that all the operators $b_{rr}(u)$, $1\leqslant r\leqslant \varkappa$, on V° commute. $\square$

Now we are ready to prove the main result of this subsection.

Theorem 4.8. *Every finite-dimensional irreducible representation V of the twisted super Yangian $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$ is a highest $\ell_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-weight representation. Moreover, V contains a unique (up to proportionality) highest $\ell_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-weight vector.*

Proof. By Lemma 4.5, V° is nontrivial. Hence it follows from Lemmas 4.6, 4.7 that V° contains a common eigenvector $\eta\neq 0$ for all operators $b_{rr}(u)$, $1\leqslant r\leqslant\varkappa$. Therefore, the vector η satisfies (4.1) for some formal series $\mu_i(u)$.

Consider the submodule $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}\eta$ in V, as V is irreducible, we conclude that $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}\eta$ coincides with V. The uniqueness of η (up to proportionality) follows from Corollary 3.4 and the weight subspaces of the operators $s_i\varepsilon_i b_{ii}^{(1)}/2$ used in the proof of Lemma 4.5. $\square$

4.2. **Verma modules.** For any $\varkappa$-tuple $\boldsymbol{\mu}(u)=(\mu_i(u))_{1\leqslant i\leqslant\varkappa}$, where $\mu_i(u)\in\varepsilon_i+u^{-1}\mathbb{C}[[u^{-1}]]$, denote by $M(\boldsymbol{\mu}(u))$ the quotient of $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$ by the left ideal generated by all coefficients of the series $b_{ij}(u)$, for $1\leqslant i<j\leqslant\varkappa$, and $b_{ii}(u)-\mu_i(u)$, for $1\leqslant i\leqslant\varkappa$. We call $M(\boldsymbol{\mu}(u))$ the *Verma module* with highest $\ell_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-weight $\boldsymbol{\mu}(u)$.

The Verma module $M(\boldsymbol{\mu}(u))$ may be trivial due to nontrivial relations. If $M(\boldsymbol{\mu}(u))$ is nontrivial, then denote by $V(\boldsymbol{\mu}(u))$ the unique irreducible quotient. Clearly, any irreducible highest $\ell_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-weight module of $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$ with highest $\ell_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-weight $\boldsymbol{\mu}(u)$ is isomorphic to $V(\boldsymbol{\mu}(u))$.

In the rest of this subsection, we discuss the sufficient and necessary condition for $M(\boldsymbol{\mu}(u))$ being nontrivial.

Before stating and proving the theorem, we prepare a few lemmas that will be useful. For each $1\leqslant i\leqslant\varkappa$, set

eq:beta

$$\beta_i(u,v)=\sum_{a=i}^{\varkappa} b_{ia}(u)b_{ai}(v). \tag{4.11}$$

Lemma 4.9. *For $1\leqslant i<\varkappa$, if $u+v=\rho_{i+1}$, then we have*

$$\begin{aligned} b_{ii}(u)b_{ii}(v) &+ \frac{1}{u-v}\sum_{a=i+1}^{\varkappa} s_a\big(b_{aa}(u)b_{ii}(v)-b_{aa}(v)b_{ii}(u)\big) \\ &\equiv b_{i+1,i+1}(u)b_{i+1,i+1}(v)+\frac{1}{u-v}\sum_{a=i+2}^{\varkappa} s_a\big(b_{aa}(u)b_{i+1,i+1}(v)-b_{aa}(v)b_{i+1,i+1}(u)\big). \end{aligned}$$

Proof. For $a > i$, it follows from (3.3) that

$$b_{ia}(u)b_{ai}(v) \equiv \frac{s_a}{u-v}\big(b_{aa}(u)b_{ii}(v) - b_{aa}(v)b_{ii}(u)\big) + \frac{s_a}{u+v}\big(\beta_i(u,v) - \beta_a(v,u)\big).$$

Summing over a, we have

eq:lem-nontrivial-1

$$\begin{aligned}\frac{u+v-\rho_{i+1}}{u+v}\beta_i(u,v) \equiv b_{ii}(u)b_{ii}(v) &- \frac{1}{u+v}\sum_{a=i+1}^{\varkappa} s_a\beta_a(v,u)\\ &+ \frac{1}{u-v}\sum_{a=i+1}^{\varkappa} s_a\big(b_{aa}(u)b_{ii}(v) - b_{aa}(v)b_{ii}(u)\big).\end{aligned} \tag{4.12}$$

Interchanging u and v and taking the difference, we obtain

eq:lem-nontrivial-2

$$\frac{u+v-\rho_{i+1}}{u+v}\big(\beta_i(u,v) - \beta_i(v,u)\big) \equiv \frac{1}{u+v}\sum_{a=i+1}^{\varkappa} s_a\big(\beta_a(u,v) - \beta_a(v,u)\big), \tag{4.13}$$

where we also used that $b_{ii}(u)b_{ii}(v) \equiv b_{ii}(v)b_{ii}(u)$, see Lemma 4.7. Note that $\beta_\varkappa(u,v) \equiv \beta_\varkappa(v,u)$, one easily shows by a reverse induction on i that $\beta_i(u,v) \equiv \beta_i(v,u)$ using (4.13).

Applying (4.12) for i and $i+1$ and using $\beta_i(u,v) \equiv \beta_i(v,u)$, one has

eq:lem-nontrivial-3

$$\begin{aligned}&\frac{u+v-\rho_{i+1}}{u+v}\big(\beta_i(u,v) - \beta_{i+1}(u,v)\big)\\ &\equiv b_{ii}(u)\,b_{ii}(v) - b_{i+1,i+1}(u)b_{i+1,i+1}(v) + \frac{1}{u-v}\sum_{a=i+1}^{\varkappa} s_a\big(b_{aa}(u)b_{ii}(v) - b_{aa}(v)b_{ii}(u)\big)\\ &\qquad - \frac{1}{u-v}\sum_{a=i+2}^{\varkappa} s_a\big(b_{aa}(u)b_{i+1,i+1}(v) - b_{aa}(v)b_{i+1,i+1}(u)\big).\end{aligned} \tag{4.14}$$

Now the statement follows immediately if $u+v = \rho_{i+1}$. □

It is convenient to set

eq:def-tl-b

$$\tilde{b}_{ii}(u) := (2u - \rho_{i+1})b_{ii}(u) + \sum_{a=i+1}^{\varkappa} s_a b_{aa}(u), \quad 1 \leqslant i \leqslant \varkappa. \tag{4.15}$$

lem:b-in-t

Lemma 4.10. *Regard $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$ as a subalgebra of $\mathcal{Y}_{\boldsymbol{s}}$ as in Proposition 3.3. Then we have*

$$\tilde{b}_{ii}(u) \approx \big(2\varepsilon_i u - \varepsilon_i\rho_{i+1} + \varpi_{i+1}\big)t_{ii}(u)t'_{ii}(-u), \tag{4.16}$$

where $\varpi_i = \sum_{j=i}^{\varkappa}\varepsilon_j s_j$ and $A \approx B$ if $A\xi = B\xi$ for any highest $\ell_{\boldsymbol{s}}$-weight vector ξ.

Proof. For $1 \leqslant i \leqslant \varkappa$, set

$$\psi_i(u) = \sum_{a=i}^{\varkappa} t_{ia}(u)t'_{ai}(-u), \quad \wp_i(u) = \sum_{a=i}^{\varkappa} t'_{ia}(-u)t_{ai}(u).$$

By (2.10) and Proposition 2.7, for $a > i$, we have

eq:inlemb-t

$$t_{ia}(u)t'_{ai}(-u) \approx \frac{s_a}{2u}(\psi_i(u) - \wp_a(u)), \quad [t_{ii}(u), t'_{ii}(-u)] \approx \frac{s_i}{2u}(\psi_i(u) - \wp_i(u)). \tag{4.17}$$

Therefore, we obtain

eq:psi

$$\psi_i(u) \approx t_{ii}(u)t'_{ii}(-u) + \sum_{a=i+1}^{\varkappa}\frac{s_a}{2u}(\psi_i(u) - \wp_a(u)), \quad 1 \leqslant i \leqslant \varkappa. \tag{4.18}$$

Similarly, one has

eq:wp

$$\wp_i(u) \approx t'_{ii}(-u)t_{ii}(u) + \sum_{a=i+1}^{\varkappa}\frac{s_a}{2u}(\wp_i(u) - \psi_a(u)), \quad 1 \leqslant i \leqslant \varkappa. \tag{4.19}$$

By Proposition 2.7, we have $t_{ii}(u)t'_{ii}(-u)\approx t'_{ii}(-u)t_{ii}(u)$ for all $1\leqslant i\leqslant \varkappa$. Also note that it follows from (4.17) that $\psi_\varkappa(u)\approx \wp_\varkappa(u)$. By a reverse induction and using (4.18), (4.19), one proves that $\psi_i(u)\approx\wp_i(u)$ for all $1\leqslant i\leqslant\varkappa$. Hence, we have

eq:inlem-b-t-wp2

$$(2u-\rho_{i+1})\wp_i(u)\approx 2ut_{ii}(u)t'_{ii}(-u)-\sum_{a=i+1}^{\varkappa}s_a\wp_a(u),\quad 1\leqslant i\leqslant\varkappa. \tag{4.20}$$

On the other hand, it follows from (3.12) and (4.17) that

eq:inlem-b-t-wp1

$$b_{ii}(u)\approx\varepsilon_i t_{ii}(u)t'_{ii}(-u)+\sum_{a=i+1}^{\varkappa}\frac{\varepsilon_a s_a}{2u}\big(\wp_i(u)-\wp_a(u)\big),\quad 1\leqslant i\leqslant\varkappa. \tag{4.21}$$

Solving $b_{ii}(u)$ in terms of $t_{jj}(u)t'_{jj}(-u)$ from the system of equations (4.20) and (4.21), it is not hard to see by a brute force computation that

$$(2u-\rho_{i+1})b_{ii}(u)+\sum_{a=i+1}^{\varkappa}s_a b_{aa}(u)\approx\big(2\varepsilon_i u-\varepsilon_i\rho_{i+1}+\varpi_{i+1}\big)t_{ii}(u)t'_{ii}(-u). \qquad \square$$

Now we are ready to prove the main theorem of this subsection.

thmnontrivial

Theorem 4.11. *The Verma module $M(\boldsymbol{\mu}(u))$ is nontrivial if and only if*

thm:nontrivial-N

$$\mu_\varkappa(u)\mu_\varkappa(-u)=1, \tag{4.22}$$

and for $1\leqslant i<\varkappa$, the following conditions are satisfied

thm:nontrivial-i

$$\tilde\mu_i(u)\tilde\mu_i(-u+\rho_{i+1})=\tilde\mu_{i+1}(u)\tilde\mu_{i+1}(-u+\rho_{i+1}), \tag{4.23}$$

where

eq:mu-tilde

$$\tilde\mu_i(u)=(2u-\rho_{i+1})\mu_i(u)+\sum_{a=i+1}^{\varkappa}s_a\mu_a(u). \tag{4.24}$$

Proof. We first show that conditions (4.22) and (4.23) are necessary. By the unitary condition (3.4), we have

$$\sum_{a=1}^{\varkappa}b_{\varkappa a}(u)b_{a\varkappa}(-u)=1.$$

Then (4.22) follows from the above equation applied to the highest weight vector of $V(\boldsymbol{\mu}(u))$. Applying Lemma 4.5 to the highest weight vector of $V(\boldsymbol{\mu}(u))$, we get

$$\mu_i(u)\mu_i(v)+\frac{1}{u-v}\sum_{a=i+1}^{\varkappa}s_a\big(\mu_a(u)\mu_i(v)-\mu_a(v)\mu_i(u)\big)$$
$$=\mu_{i+1}(u)\mu_{i+1}(v)+\frac{1}{u-v}\sum_{a=i+2}^{\varkappa}s_a\big(\mu_a(u)\mu_{i+1}(v)-\mu_a(v)\mu_{i+1}(u)\big),$$

where $u+v=\rho_{i+1}$. It is not hard to see that the above equation is equivalent to conditions (4.23).

Conversely, suppose the conditions (4.22) and (4.23) are satisfied. We shall show that there exists a highest $\ell_{\boldsymbol{s}}$-weight vector ξ of highest $\ell_{\boldsymbol{s}}$-weight $\boldsymbol{\lambda}(u)$ such that ξ is of highest $\ell_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-weight $\boldsymbol{\mu}(u)$. This proves that the Verma module $M(\boldsymbol{\mu}(u))$ is non-trivial.

First, observe from Lemma 2.8 and (3.12) that $b_{ij}(u)\xi=0$ for $1\leqslant i<j\leqslant\varkappa$.

We construct $\lambda_i(u)\in 1+u^{-1}\mathbb{C}[[u^{-1}]]$ inductively as follows. By (4.22), there exists $\lambda_\varkappa(u)\in 1+u^{-1}\mathbb{C}[[u^{-1}]]$ such that

$$\mu_\varkappa(u)=\varepsilon_\varkappa\lambda_\varkappa(u)/\lambda_\varkappa(-u).$$

Suppose we already have $\lambda_j(u)$ for $i<j\leqslant\varkappa$. Define

$$\Lambda_i(u)=\frac{(2\varepsilon_{i+1}u-\varepsilon_{i+1}\rho_{i+2}+\varpi_{i+2})\tilde\mu_i(u)\lambda_{i+1}(u)}{(2\varepsilon_i u-\varepsilon_i\rho_{i+1}+\varpi_{i+1})\tilde\mu_{i+1}(u)\lambda_{i+1}(-u+\rho_{i+1})}.$$

Note that if $\varepsilon_i=\varepsilon_{i+1}$, then we have

eq:good1

$$2\varepsilon_{i+1}u-\varepsilon_{i+1}\rho_{i+2}+\varpi_{i+2}=2\varepsilon_i u-\varepsilon_i\rho_{i+1}+\varpi_{i+1};\tag{4.25}$$

if $\varepsilon_i=-\varepsilon_{i+1}$, then we have

eq:good2

$$2\varepsilon_{i+1}u-\varepsilon_{i+1}\rho_{i+2}+\varpi_{i+2}=2\varepsilon_i(-u+\rho_{i+1})-\varepsilon_i\rho_{i+1}+\varpi_{i+1}.\tag{4.26}$$

Therefore, one easily checks that the condition (4.23) ensures that $\Lambda_i(u)\Lambda_i(-u+\rho_{i+1})=1$. Hence there exists $\lambda_i(u)\in 1+u^{-1}\mathbb{C}[[u^{-1}]]$ such that $\Lambda_i(u)=\lambda_i(u)/\lambda_i(-u+\rho_{i+1})$.

With our choice of $\boldsymbol{\lambda}(u)$, we have

$$\frac{\tilde\mu_i(u)}{\tilde\mu_{i+1}(u)}=\frac{(2\varepsilon_i u-\varepsilon_i\rho_{i+1}+\varpi_{i+1})\lambda_i(u)\lambda_{i+1}(-u+\rho_{i+1})}{(2\varepsilon_{i+1}u-\varepsilon_{i+1}\rho_{i+2}+\varpi_{i+2})\lambda_{i+1}(u)\lambda_i(-u+\rho_{i+1})}$$

and $\mu_\varkappa(u)=\varepsilon_\varkappa\lambda_\varkappa(u)/\lambda_\varkappa(-u)$. By Propositions 2.7, 2.9 and Lemma 4.10, one verifies that ξ is indeed of highest $\ell_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-weight $\boldsymbol{\mu}(u)$. $\square$

4.3. **Tensor product of representations.** Recall from Proposition 3.5 that $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$ is a left coideal subalgebra in $\mathcal{Y}_{\boldsymbol{s}}$. Given a $\mathcal{Y}_{\boldsymbol{s}}$-module L and a $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-module V, then $L\otimes V$ is a $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-module given by the coproduct formula (3.16) in Proposition 3.5.

Let $L=L(\boldsymbol{\lambda}(u))$ be a highest $\ell_{\boldsymbol{s}}$-weight module over $\mathcal{Y}_{\boldsymbol{s}}$ with a highest $\ell_{\boldsymbol{s}}$-weight vector ξ. Let $V=V(\boldsymbol{\mu}(u))$ be a highest $\ell_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-weight module over $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$ with a highest $\ell_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-weight vector η. We end this section by showing that $\xi\otimes\eta$ is a highest $\ell_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-weight vector and calculating its highest $\ell_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-weight.

Again we shall use the convenient notation $A\approx B$ if $A\xi=B\xi$.

lem:sum-T-

Lemma 4.12. *For $1\leqslant i<a\leqslant\varkappa$, we have*

$$(2u-\rho_{i+1})t_{ia}(u)t'_{ai}(-u)+\sum_{c=i+1}^{a}s_ct_{ca}(u)t'_{ac}(-u)\approx s_at_{ii}(u)t'_{ii}(-u).$$

Proof. By (4.17), one obtains

$$\begin{aligned}(2u-\rho_{i+1})t_{ia}(u)t'_{ai}(-u)+&\sum_{c=i+1}^{a-1}s_ct_{ca}(u)t'_{ac}(-u)\\ &\approx\frac{s_a}{2u}\Big((2u-\rho_{i+1})(\wp_i(u)-\wp_a(u))+\sum_{c=i+1}^{a}s_c(\wp_c(u)-\wp_a(u))\Big)\\ &\approx\frac{s_a}{2u}\Big((2u-\rho_{i+1})\wp_i(u)-(2u-\rho_{a+1})\wp_a(u)+\sum_{c=i+1}^{a}s_c\wp_c(u)\Big)\\ &\approx\frac{s_a}{2u}\Big(2ut_{ii}(u)t'_{ii}(-u)-2ut_{aa}(u)t'_{aa}(-u)\Big),\end{aligned}$$

where the last equality follows from (4.20). Now the lemma follows. $\square$

prop:tensor-product

Proposition 4.13. *We have $b_{ij}(u)(\xi\otimes\eta)=0$, $1\leqslant i<j\leqslant\varkappa$, and*

$$\tilde b_{ii}(u)(\xi\otimes\eta)=\lambda_i(u)\lambda'_i(-u)\tilde\mu_i(u)(\xi\otimes\eta),\qquad 1\leqslant i\leqslant\varkappa,$$

where $\lambda'_i(u)$ and $\tilde\mu_i(u)$ are defined in (2.24) and (4.24), respectively.

Proof. It is easily seen from Lemma 2.8 and (3.16) that $b_{ij}(u)(\xi\otimes\eta)=0$.

Again we write $A\approx B$ if $A(\xi\otimes\eta)=B(\xi\otimes\eta)$. It follows from Lemma 2.8 that

$$\Delta(b_{ii}(u))\approx\sum_{a=i}^{\varkappa}\big(t_{ia}(u)t'_{ai}(-u)\otimes b_{aa}(u)\big)(\xi\otimes\eta).$$

Therefore,

$$\begin{aligned}\Delta(\tilde b_{ii}(u))&\approx(2u-\rho_{i+1})\sum_{a=i}^{\varkappa}t_{ia}(u)t'_{ai}(-u)\otimes b_{aa}(u)+\sum_{a=i+1}^{\varkappa}s_a\sum_{c=a}^{\varkappa}t_{ac}(u)t'_{ca}(-u)\otimes b_{cc}(u)\\&=\sum_{a=i}^{\varkappa}\Big((2u-\rho_{i+1})t_{ia}(u)t'_{ai}(-u)+\sum_{c=i+1}^{a}s_ct_{ca}(u)t'_{ac}(-u)\Big)\otimes b_{aa}(u)\\&\approx(2u-\rho_{i+1})t_{ii}(u)t'_{ii}(-u)\otimes b_{ii}(u)+\sum_{a=i+1}^{\varkappa}s_at_{ii}(u)t'_{ii}(-u)\otimes b_{aa}(u)\\&=t_{ii}(u)t'_{ii}(-u)\otimes\Big((2u-\rho_{i+1})b_{ii}(u)+\sum_{a=i+1}^{\varkappa}s_ab_{aa}(u)\Big)=t_{ii}(u)t'_{ii}(-u)\otimes\tilde b_{ii}(u).\end{aligned}$$

Here we applied Lemma 4.12 in the third equality. Now the statement follows. ☐

eg:tensor-1

Example 4.14. Recall the one dimensional $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-module $\mathbb{C}_\gamma=\mathbb{C}\eta_\gamma$ from Example 4.2 for $\gamma\in\mathbb{C}$. Let $L=L(\boldsymbol{\lambda}(u))$ be a highest $\ell_{\boldsymbol{s}}$-weight module over $\mathscr{Y}_{\boldsymbol{s}}$ with a highest $\ell_{\boldsymbol{s}}$-weight vector ξ. Then by Proposition 4.13 we have

$$\tilde b_{ii}(u)(\xi\otimes\eta_\gamma)=\frac{(2\varepsilon_iu-\varepsilon_i\rho_{i+1}+\varpi_{i+1}+2\gamma)u}{u-\gamma}\lambda_i(u)\lambda'_i(-u)(\xi\otimes\eta_\gamma),$$

cf. Lemma 4.10. ☐

5. Classifications in rank 1

sec:rank1

In this section, we investigate the finite-dimensional irreducible representations of twisted super Yangian of the small rank case $\varkappa=2$.

5.1. **Non-super case.** The case $\boldsymbol{s}=(1,1)$ has already been studied in [MR02, Propositions 4.4, 4.5] via identifying $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$ with (Olshanski's) twisted Yangians $\mathscr{Y}(\mathfrak{sp}_2)$ and $\mathscr{Y}(\mathfrak{so}_2)$ of types AI and AII.

prop:iff-even

Proposition 5.1 ([MR02]). *Suppose $\boldsymbol{s}=(1,1)$.*

(1) *If $\boldsymbol{\varepsilon}=(1,1)$, then the $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-module $V(\boldsymbol{\mu}(u))$ is finite-dimensional if and only if there exists a monic polynomial $P(u)$ such that $P(-u+2)=P(u)$ and*

$$\frac{\tilde\mu_1(u)}{\tilde\mu_2(u)}=\frac{P(u+1)}{P(u)}.$$

(2) *If $\boldsymbol{\varepsilon}=(1,-1)$, then the $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-module $V(\boldsymbol{\mu}(u))$ is finite-dimensional if and only if there exist $\gamma\in\mathbb{C}$ and a monic polynomial $P(u)$ such that $P(-u+2)=P(u)$, $P(\gamma)\neq0$, and*

$$\frac{\tilde\mu_1(u)}{\tilde\mu_2(u)}=\frac{P(u+1)}{P(u)}\cdot\frac{\gamma-u}{\gamma+u-1}.$$

In this case, the pair $(P(u),\gamma)$ is unique.

prop:con-even

Proposition 5.2 ([MR02]). *Suppose $\boldsymbol{s}=(1,1)$.*

(1) *If $\boldsymbol{\varepsilon}=(1,1)$, then any finite-dimensional irreducible $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-module $V(\boldsymbol{\mu}(u))$ is isomorphic to the restriction of a $\mathscr{Y}(\mathfrak{gl}_2)$-module L, where L is some finite-dimensional irreducible $\mathscr{Y}(\mathfrak{gl}_2)$-module.*

(2) *If* $\boldsymbol{\varepsilon} = (1, -1)$*, then any finite-dimensional irreducible* $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$*-module* $V(\boldsymbol{\mu}(u))$ *is isomorphic to* $L \otimes \mathbb{C}_\gamma$*, where* L *is some finite-dimensional irreducible* $\mathcal{Y}(\mathfrak{gl}_2)$*-module and* $\mathbb{C}_\gamma$ *is some one-dimensional* $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$*-module defined in Example 4.2 with* $\gamma \in \mathbb{C}$.

5.2. **Super case.** In the rest of this section, we establish super analogous results to the previous propositions when $\boldsymbol{s} = (1, -1)$ or $(-1, 1)$. Our main results in this subsection are the following.

prop:rank1

Proposition 5.3. *If* $\boldsymbol{s}$ *is such that* $s_1 \neq s_2$*, then the* $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$*-module* $V(\boldsymbol{\mu}(u))$ *is finite-dimensional if and only if there exists a monic polynomial* $P(u)$ *such that*

$$\frac{\tilde{\mu}_1(u)}{\tilde{\mu}_2(u)} = \varepsilon_1\varepsilon_2(-1)^{\deg P}\frac{P(u)}{P(-u+s_2)}.$$

Proof. The $\Longleftarrow$ part follows from Theorem 2.4 and Theorem 6.1 below as $V(\boldsymbol{\mu}(u))$ can be obtained as a quotient of the restriction of a finite-dimensional irreducible $\mathcal{Y}_{\boldsymbol{s}}$-module.

To show the $\Longrightarrow$ part, note that due to the condition (4.23), such a polynomial $P(u)$ exists provided that $\tilde{\mu}_1(u)/\tilde{\mu}_2(u)$ or alternatively $\mu_1(u)/\mu_2(u)$ is an expansion of a rational function in u at $u = \infty$.

We will work on

$$x_{ij}(u) = \varepsilon_i\delta_{ij} + \sum_{r>0} x_{ij}^{(r)}u^{-r} := b_{ij}(u + s_2/2)$$

and the case $\boldsymbol{s} = (1, -1)$ since the general case $\boldsymbol{s} = (-1, 1)$ is similar by inserting the signs at suitable positions or using certain isomorphisms.

Using (3.3), we have

$$[b_{21}(u), b_{22}(v)] = \frac{-1}{u-v}(b_{21}(u)b_{22}(v) - b_{21}(v)b_{22}(u)) + \frac{1}{u+v}(b_{21}(v)b_{11}(u) + b_{22}(v)b_{21}(u)),$$

which gives

$$\begin{aligned}\frac{u+v+1}{u+v}b_{22}(v)b_{21}(u) = b_{21}(u)b_{22}(v) &- \frac{1}{u+v}b_{21}(v)b_{11}(u)\\ &- \frac{1}{u-v}(b_{21}(v)b_{22}(u) - b_{21}(u)b_{22}(v)).\end{aligned}$$

Substituting $u \to u - 1/2$, $v \to v - 1/2$ and dividing both sides by $(u+v)/(u+v-1)$, we obtain

eq:new-b

$$\begin{aligned}x_{22}(v)x_{21}(u) = {}& x_{21}(u)x_{22}(v) - \frac{1}{v+u}(x_{21}(v)x_{11}(u) + x_{21}(u)x_{22}(v))\\ &+ \frac{1}{v-u}(x_{21}(v)x_{22}(u) - x_{21}(u)x_{22}(v)) + \frac{1}{v^2-u^2}(x_{21}(u)x_{22}(v) - x_{21}(v)x_{22}(u)).\end{aligned} \tag{5.1}$$

Taking the coefficients of $u^{-k}v^{-2}$ and v^{-2}, we have

eq:k2

$$x_{22}^{(2)}x_{21}^{(k)} = x_{21}^{(k)}(x_{22}^{(2)} - 2x_{22}^{(1)} + x_{22}^{(0)}) - x_{21}^{(1)}(x_{11}^{(k)} - x_{22}^{(k)}), \tag{5.2}$$

eq:just2

$$x_{22}^{(2)}x_{21}(u) = x_{21}(u)(x_{22}^{(2)} - 2x_{22}^{(1)} + x_{22}^{(0)}) - x_{21}^{(1)}(x_{11}(u) - x_{22}(u)). \tag{5.3}$$

Similarly, taking the coefficients of $u^{-k}v^{-3}$ and v^{-3}, we have

eq:k3

$$\begin{aligned}x_{22}^{(3)}x_{21}^{(k)} = {}& -2\varepsilon_2 x_{21}^{(k+2)} + x_{21}^{(k)}(x_{22}^{(3)} - 2x_{22}^{(2)} + x_{22}^{(1)})\\ &- x_{21}^{(2)}(x_{11}^{(k)} - x_{22}^{(k)}) + x_{21}^{(1)}(x_{11}^{(k+1)} + x_{22}^{(k+1)} - x_{22}^{(k)}),\end{aligned} \tag{5.4}$$

eq:just3

$$\begin{aligned}x_{22}^{(3)}x_{21}(u) = {}& -2\varepsilon_2 u^2 x_{21}(u) + x_{21}(u)(x_{22}^{(3)} - 2x_{22}^{(2)} + x_{22}^{(1)})\\ &- x_{21}^{(2)}(x_{11}(u) - x_{22}(u)) + x_{21}^{(1)}(ux_{11}(u) + ux_{22}(u) - x_{22}(u)).\end{aligned} \tag{5.5}$$

Denote by η the highest $\ell_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-weight vector of $V(\boldsymbol{\mu}(u))$. Define the series $\lambda_i(u)$ by

$$\lambda_i(u) = \sum_{r \geqslant 0}\lambda_{ir}u^{-r}, \qquad i = 1, 2,$$

where $\lambda_{ir}\in\mathbb{C}$ satisfies $x_{ii}^{(r)}\eta=\lambda_{ir}\eta$. In particular, $\lambda_{i0}=\varepsilon_i$.

(1) The case $\varepsilon_1=\varepsilon_2$. We first prove that for $r>0$, the vector $x_{21}^{(2r)}\eta$ is a linear combination of vectors $x_{21}^{(1)}\eta, x_{21}^{(3)}\eta,\cdots,x_{21}^{(2r-1)}\eta$.

We prove it by induction on k. Setting $k=0$ in (5.4) and noticing that $x_{21}^{(0)}=0$, we have

eq:ind-base

$$(\varepsilon_1+\varepsilon_2)x_{21}^{(2)}=x_{21}^{(1)}(x_{11}^{(1)}+x_{22}^{(1)}-\varepsilon_2). \tag{5.6}$$

Therefore, $2\varepsilon_2 x_{21}^{(2)}\eta=(\lambda_{11}+\lambda_{21}-\varepsilon_2)x_{21}^{(1)}\eta$. Now suppose that

eq:ind-hypo

$$x_{21}^{(2r)}\eta=c_1x_{21}^{(1)}\eta+\cdots+c_{2r-1}x_{21}^{(2r-1)}\eta \tag{5.7}$$

for some $c_1,\cdots,c_{2r-1}\in\mathbb{C}$. Then applying (5.3) to $x_{22}^{(3)}(c_1x_{21}^{(1)}+\cdots+c_{2k-1}x_{21}^{(2r-1)})\eta$ and using (5.6), we find that $x_{22}^{(3)}(c_1x_{21}^{(1)}+\cdots+c_{2r-1}x_{21}^{(2r-1)})\eta$ is a linear combination of vectors $x_{21}^{(1)}\eta$, $x_{21}^{(2)}\eta$, $\cdots$, $x_{21}^{(2r+1)}\eta$, which is also a linear combination of vectors $x_{21}^{(1)}\eta$, $x_{21}^{(3)}\eta$, $\cdots$, $x_{21}^{(2r-1)}\eta$ by (5.6) and (5.7). Similarly, by (5.3), $x_{22}^{(3)}x_{21}^{(2r)}\eta$ is equal to $-2\varepsilon_2x_{21}^{(2r+2)}\eta$ plus a linear combination of vectors $x_{21}^{(1)}\eta$, $x_{21}^{(3)}\eta$. Thus, the claim is proved.

Let $\eta_r=x_{21}^{(2r-1)}\eta$ for $r\in\mathbb{Z}_{>0}$. Since $V(\boldsymbol{\mu}(u))$ is finite-dimensional, there exists a minimal nonnegative integer k be the minimal nonnegative integer such that η_{k+1} is a linear combination of the vectors $\eta_1,\cdots,\eta_k$,

eq:pf-low-0

$$\eta_{k+1}=c_1\eta_1+\cdots+c_k\eta_k. \tag{5.8}$$

Then for any $r>k$, one proves similarly as in the proof of the above claim that

$$\eta_r=a_{r1}\eta_1+\cdots+a_{rk}\eta_k$$

for some $a_{ri}\in\mathbb{C}$, where $1\leqslant i\leqslant k$. Therefore, there exist series $a_i(u)\in u^{1-2i}(1+\mathbb{C}[[u^{-1}]])$, $1\leqslant i\leqslant k$, in u^{-1} such that

eq:x21-eta

$$x_{21}(u)\eta=a_1(u)\eta_1+a_2(u)\eta_2+\cdots+a_k(u)\eta_k. \tag{5.9}$$

To simplify the notation, we use the following shorthand notation for these scalars,

$$\Lambda_0=\lambda_{22}-2\lambda_{21}+\varepsilon_2,\qquad \Lambda_1=\lambda_{23}-2\lambda_{22}+\lambda_{21},$$
$$\beta_r=\lambda_{2r}-\lambda_{1r},\qquad \theta_r=\lambda_{1,r+1}+\lambda_{2,r+1}-\lambda_{2r}.$$

By (5.3) and (5.9), we have

eq:pf-low-1

$$\begin{aligned} x_{22}^{(2)}x_{21}(u)\eta &=\Lambda_0x_{21}(u)\eta-(\lambda_1(u)-\lambda_2(u))\eta_1\\ &=\Lambda_0(a_1(u)\eta_1+a_2(u)\eta_2+\cdots+a_k(u)\eta_k)-(\lambda_1(u)-\lambda_2(u))\eta_1. \end{aligned}\tag{5.10}$$

On the other hand, by (5.2), we have

eq:pf-low-2

$$x_{22}^{(2)}\sum_{r=1}^{k}a_r(u)\eta_k=\sum_{r=1}^{k}a_r(u)x_{22}^{(2)}x_{21}^{(2r-1)}\eta=\sum_{r=1}^{k}a_r(u)(\Lambda_0\eta_r+\beta_{2r-1}\eta_1). \tag{5.11}$$

Comparing the coefficients of η_1, it follows that

eq:pf-low-3

$$\lambda_2(u)-\lambda_1(u)=\sum_{k=1}^{r}\beta_{2k-1}a_k(u). \tag{5.12}$$

Recall that (5.6) implies $x_{21}^{(2)}\eta=\frac{1}{2}\theta_0\varepsilon_2\eta_1$. Similarly, by (5.5), we have

$$\begin{aligned} &x_{22}^{(3)}x_{21}(u)\eta\\ &=(-2\varepsilon_2u^2+\Lambda_1)x_{21}(u)\eta+\Big(u\lambda_1(u)+u\lambda_2(u)-\lambda_2(u)+\frac{1}{2}\theta_0\varepsilon_2(\lambda_2(u)-\lambda_1(u))\Big)\eta_1\\ &=(-2\varepsilon_2u^2+\Lambda_1)\sum_{r=1}^{k}a_r(u)\eta_r+\Big(u\lambda_1(u)+u\lambda_2(u)-\lambda_2(u)+\frac{1}{2}\theta_0\varepsilon_2(\lambda_2(u)-\lambda_1(u))\Big)\eta_1. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} x_{22}^{(3)}\sum_{r=1}^{k} a_r(u)\eta_r &= \sum_{r=1}^{k} a_r(u) x_{22}^{(3)} x_{21}^{(2r-1)}\eta \\ &= \sum_{r=1}^{k} a_r(u)\Big(-2\varepsilon_2\eta_{r+1} + \Lambda_1\eta_r + \frac{1}{2}\theta_0\varepsilon_2\beta_{2r-1}\eta_1 + \theta_{2r-1}\eta_1\Big). \end{aligned}$$

Applying (5.8) to the above equality and comparing the coefficients of η_r for $1 < r \leqslant k$, we obtain that

$$(-2\varepsilon_2 u^2 + \Lambda_1)a_r(u) = (-2\varepsilon_2 a_{r-1}(u) + \Lambda_1 a_r(u) - 2\varepsilon_2 c_r a_k(u)),$$

which reduces to

eq:pf-low-4

$$a_{r-1}(u) = u^2 a_r(u) - c_r a_k(u). \tag{5.13}$$

Hence, for any $1 \leqslant r \leqslant k$, we have $a_r(u) = \mathfrak{P}_r(u)a_k(u)$, where $\mathfrak{P}_r(u)$ is a polynomial in u of degree $2(k-r)$. Finally, taking the coefficients of η_1 and using (5.13), we conclude that

eq:pf-low-5

$$\Big(u - \frac{1}{2}\theta_0\varepsilon_2\Big)\lambda_1(u) + \Big(u + \frac{1}{2}\theta_0\varepsilon_2 - 1\Big)\lambda_2(u) = \mathfrak{P}(u)a_k(u), \tag{5.14}$$

where $\mathfrak{P}(u)$ is a polynomial in u of degree $2k$. Note that (5.12) and (5.13) imply

eq:pf-low-6

$$\lambda_2(u) - \lambda_1(u) = \mathcal{P}(u)a_k(u), \tag{5.15}$$

where $\mathcal{P}(u)$ is a polynomial in u of degree at most $2k-2$. It follows from (5.14) and (5.15) that $\lambda_1(u)/\lambda_2(u)$ is an expansion of a rational function in u at $u=\infty$, completing the proof for the case $\varepsilon_1 = \varepsilon_2$.

(2) The case $\varepsilon_1 \neq \varepsilon_2$. The proof for this case is very similar to that of the previous case. The difference is that one needs to use $x_{21}^{(2r)}\eta$ and $x_{22}^{(4)}x_{21}^{(2r)}$ instead of $x_{21}^{(2r-1)}\eta$ and $x_{22}^{(2)}x_{21}^{(2r-1)}$, respectively, cf. [Mol98, Proposition 6.1] as we have $x_{21}^{(1)} = 0$ in this case. Then (5.15) is replaced with

$$(u^2 + \cdots)\lambda_2(u) - (u^2 + \cdots)\lambda_1(u) = \mathcal{P}(u)a_k(u),$$

where $\cdots$ stand for two different linear polynomials in u and $\mathcal{P}(u)$ is a polynomial in u of degree $2k+2$. Note that in this case $\mathfrak{P}(u)$ in (5.14) is of degree at most $2k$. We omit the detail for this case. □

5.3. **Preparations.** In this subsection, we prepare ingredients to establish super analogue of Proposition 5.2 and always assume that $\boldsymbol{s} = (s_1, s_2)$ satisfies $s_1 \neq s_2$.

We start with simple calculations for 2-dimensional evaluation modules. Let $\mathscr{a}$ and $\mathscr{b}$ be complex numbers such that $\mathscr{a} + \mathscr{b} \neq 0$. Then the evaluation $\mathscr{Y}_{\boldsymbol{s}}$-module $L(\mathscr{a}, \mathscr{b})$ is two dimensional. Let v^+ be a nonzero singular vector and set $v^- = e_{21}v^+$.

lemn2

Lemma 5.4. *We have the following explicit action,*

$$\begin{aligned}
&t_{11}(u)v^+ = \frac{u + s_1\mathscr{a}}{u}v^+, \qquad t_{12}(u)v^+ = 0,\\
&t_{22}(u)v^+ = \frac{u - s_1\mathscr{b}}{u}v^+, \qquad t_{21}(u)v^+ = -\frac{s_1}{u}v^-,\\
&t_{11}(u)v^- = \frac{u - s_1 + s_1\mathscr{a}}{u}v^-, \qquad t_{21}(u)v^- = 0,\\
&t_{22}(u)v^- = \frac{u - s_1 - s_1\mathscr{b}}{u}v^-, \qquad t_{12}(u)v^- = \frac{s_1(\mathscr{a}+\mathscr{b})}{u}v^+,\\
&t'_{11}(u)v^+ = \frac{u(u - s_1 - s_1\mathscr{b})}{(u - s_1 + s_1\mathscr{a})(u - s_1\mathscr{b})}v^+, \qquad t'_{12}(u)v^+ = 0,\\
&t'_{22}(u)v^+ = \frac{u}{u - s_1\mathscr{b}}v^+, \qquad t'_{21}(u)v^+ = \frac{s_1 u}{(u - s_1 + s_1\mathscr{a})(u - s_1\mathscr{b})}v^-,\\
&t'_{11}(u)v^- = \frac{u}{u - s_1 + s_1\mathscr{a}}v^+, \qquad t'_{12}(u)v^- = \frac{-s_1(\mathscr{a}+\mathscr{b})u}{(u - s_1 + s_1\mathscr{a})(u - s_1\mathscr{b})}v^+,
\end{aligned}$$

$$t'_{22}(u)v^- = \frac{u(u+s_1\mathscr{a})}{(u-s_1+s_1\mathscr{a})(u-s_1\mathscr{b})}v^-, \qquad t'_{21}(u)v^- = 0.$$

In particular, $(u+s_1-s_1\mathscr{a})(u+s_1\mathscr{b})t_{ij}(u)t'_{kl}(-u)$ *and* $(u+s_1-s_1\mathscr{a})(u+s_1\mathscr{b})b_{ij}(u)$ *act on* $L(\mathscr{a},\mathscr{b})$ *polynomially in* u.

Proof. The formulas follow from straightforward computation and the second statement follows from the formulas. □

We call two $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-modules V_1, V_2 are *almost isomorphic* if V_1 is isomorphic to the module obtained by pulling back V_2 through an automorphism of the form $\mathscr{M}_{\hbar(u)}$, see (3.10). In particular, the modules $V(\boldsymbol{\mu}(u))$ and $V(\boldsymbol{\nu}(u))$ are almost isomorphic if and only if

$$\frac{\tilde{\mu}_i(u)}{\tilde{\mu}_{i+1}(u)} = \frac{\tilde{\nu}_i(u)}{\tilde{\nu}_{i+1}(u)}, \qquad 1 \leqslant i < \varkappa.$$

Similarly, one can define *almost isomorphic* $\mathcal{Y}_{\boldsymbol{s}}$-modules. Then the modules $L(\boldsymbol{\lambda}(u))$ and $L(\boldsymbol{\Lambda}(u))$ are almost isomorphic if and only if

$$\frac{\lambda_i(u)}{\lambda_{i+1}(u)} = \frac{\Lambda_i(u)}{\Lambda_{i+1}(u)}, \qquad 1 \leqslant i < \varkappa.$$

If V_1, V_2 are *almost isomorphic*, then we write $V_1 \simeq V_2$.

To understand the module structure of finite dimensional irreducible $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$ modules, it suffices to investigate these modules up to almost isomorphism.

We shall also need the dual modules. Let L be a finite-dimensional $\mathcal{Y}_{\boldsymbol{s}}$-module. The *dual* L^* of L is the representation of $\mathcal{Y}_{\boldsymbol{s}}$ on the dual vector space of L defined as follows:

$$(y\cdot\omega)(v) := (-1)^{|\omega||y|}\omega(\Omega(y)\cdot v), \quad y\in\mathcal{Y}_{\boldsymbol{s}},\ \omega\in L^*,\ v\in L,$$

where Ω is defined in (2.13). Let w be another finite-dimensional $\mathcal{Y}_{\boldsymbol{s}}$-module. Then we have $(L\otimes W)^* = L^*\otimes W^*$. Let L be a finite-dimensional $\mathcal{Y}_{\boldsymbol{s}}$-module of highest $\ell_{\boldsymbol{s}}$-weight generated by a highest $\ell_{\boldsymbol{s}}$-weight vector ζ. Let $\zeta^*\in L^*$ be the vector such that $\zeta^*(\zeta)=1$ and $\zeta^*(v)=0$ for all $v\in L$ with $\mathrm{wt}(v)\neq\mathrm{wt}(\zeta)$.

`cor dual weight`

Corollary 5.5. *Let V be a finite-dimensional $\mathcal{Y}_{\boldsymbol{s}}$-module of highest $\ell_{\boldsymbol{s}}$-weight generated by a highest $\ell_{\boldsymbol{s}}$-weight vector v of $\ell_{\boldsymbol{s}}$-weight $\boldsymbol{\zeta}(u) = (\zeta_i(u))_{1\leqslant i\leqslant\varkappa}$. Then v^* is of $\ell_{\boldsymbol{s}}$-weight $\widehat{\boldsymbol{\zeta}}(u) = (\widehat{\zeta}_i(u))_{1\leqslant i\leqslant\varkappa}$, where*

$$\widehat{\zeta}_i(u) = \frac{1}{\lambda_i(-u+\rho_{i+1})}\prod_{k=i+1}^{\varkappa}\frac{\lambda_k(-u+\rho_k)}{\lambda_k(-u+\rho_{k+1})}.$$

Note that

$$\begin{aligned}\Omega(b_{ij}(u)) &= \sum_{a=1}^{\varkappa}\varepsilon_a(-1)^{|i||a|+|a|+|a||j|+|j|+(|a|+|i|)(|a|+|j|)}t_{ja}(u)t'_{ai}(-u)\\ &= \sum_{a=1}^{\varkappa}\varepsilon_a(-1)^{|i||j|+|j|}t_{ja}(u)t'_{ai}(-u) = (-1)^{|i||j|+|j|}b_{ji}(u).\end{aligned}$$

This means that the subalgebra $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$ of $\mathcal{Y}_{\boldsymbol{s}}$ is stable under Ω and the restriction of Ω to $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$ yields an anti-automorphism of $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$.

Let V be a finite-dimensional $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-module. The *dual* V^* of V is the representation of $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$ on the dual vector space of V^* defined as follows:

$$(y\cdot\omega)(v) := (-1)^{|\omega||y|}\omega(\Omega(y)\cdot v), \quad y\in\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}},\ \omega\in V^*,\ v\in V.$$

Clearly, the dual $\mathbb{C}_\gamma^*$ of the one-dimensional module $\mathbb{C}_\gamma$ is isomorphic to $\mathbb{C}_\gamma$.

Let L be a finite-dimensional $\mathcal{Y}_{\boldsymbol{s}}$-module and V a finite-dimensional $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-module, then it is straightforward to verify that

$$(L\otimes V)^* \cong L^*\otimes V^*.$$

Fix $k \in \mathbb{Z}_{>0}$. Let (a_i, b_i) be a pair of complex numbers for each $1 \leqslant i \leqslant k$. Consider the following tensor product of evaluation $\mathscr{Y}_{\boldsymbol{s}}$-modules,

$$L(\boldsymbol{a}, \boldsymbol{b}) = L(a_1, b_1) \otimes \cdots \otimes L(a_k, b_k).$$

lem:dual

Lemma 5.6. *We have*

$$L(\boldsymbol{a}, \boldsymbol{b})^* \simeq L(b_1 + 1, a_1 - 1) \otimes \cdots \otimes L(b_k + 1, a_k - 1)$$

as $\mathscr{Y}_{\boldsymbol{s}}$-modules. Moreover, we have

$$\big(L(\boldsymbol{a}, \boldsymbol{b}) \otimes \mathbb{C}_\gamma\big)^* \simeq L(b_1 + 1, a_1 - 1) \otimes \cdots \otimes L(b_k + 1, a_k - 1) \otimes \mathbb{C}_\gamma$$

as $\mathscr{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-modules.

Proof. The lemma follows from Corollary 5.5 by a direct computation. □

Though we work with the case $\varkappa = 2$, the dual modules can be generalized to arbitrary $\varkappa$ and $\boldsymbol{s}$.

5.4. **The case $\varepsilon_1 = \varepsilon_2$.** Now we assume further that $\varepsilon_1 = \varepsilon_2$. Suppose $V(\boldsymbol{\mu}(u))$ is finite dimensional, then by Proposition 5.3 we have

P-def
$$\frac{\tilde{\mu}_1(u)}{\tilde{\mu}_2(u)} = (-1)^{\deg P(u)} \frac{P(u)}{P(-u - s_1)}. \tag{5.16}$$

We assume further that $P(u)$ and $P(-u - s_1)$ are relatively prime. Otherwise, we may cancel common factors and obtain a polynomial of smaller degree. Then $P(-s_1/2) \neq 0$. Suppose

$$P(u) = (u + s_1 a_1)(u + s_1 a_2) \cdots (u + s_1 a_l)$$

where $l = \deg P(u)$ and $a_i \in \mathbb{C}$ for $1 \leqslant i \leqslant l$. We have $a_i \neq 1/2$.

Set $k = \lfloor \frac{l+1}{2} \rfloor$. Introduce k pair of complex numbers (a_i, b_i), where a_i are defined as above while b_i are defined as follows,

- if l is even, then $b_i = a_{i+k} - 1$ for $1 \leqslant i \leqslant k$;
- if l is odd, then $b_i = a_{i+k} - 1$ for $1 \leqslant i < k$ and $b_k = -\frac{1}{2}$.

Then we have

P-factor
$$\frac{P(u)}{P(-u - s_1)} = (-1)^{\deg P(u)} \prod_{i=1}^{k} \frac{(u + s_1 a_i)(u + s_1 + s_1 b_i)}{(u - s_1 b_i)(u + s_1 - s_1 a_i)}. \tag{5.17}$$

The only possible cancellation is when l is odd, then

$$u + s_1 + s_1 b_k = u - s_1 b_k.$$

Note that we have

notequalto
$$a_i + b_i \neq 0, \quad a_i + b_j \neq 0, \quad a_i + a_j \neq 1, \quad b_i + b_j \neq 1 \tag{5.18}$$

for all $1 \leqslant i \neq j \leqslant k$.

We consider the following tensor product of evaluation $\mathscr{Y}_{\boldsymbol{s}}$-modules,

$$L(\boldsymbol{a}, \boldsymbol{b}) = L(a_1, b_1) \otimes \cdots \otimes L(a_k, b_k).$$

For each $1 \leqslant i \leqslant k$, $L(a_i, b_i)$ is two dimensional. We set v_i^+ to be one of its nonzero singular vector and $v_i^- = e_{21} v_i^+$. Moreover, we assume that v_i^+ are even. We also set $v^+ = v_1^+ \otimes \cdots \otimes v_k^+$. We regard $L(\boldsymbol{a}, \boldsymbol{b})$ as a $\mathscr{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-module by restriction.

prop=

Proposition 5.7. *If $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2)$ satisfies $\varepsilon_1 \neq \varepsilon_2$, then the $\mathscr{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-module $L(\boldsymbol{a}, \boldsymbol{b})$ is irreducible. Moreover, the finite-dimensional irreducible $\mathscr{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-module $V(\boldsymbol{\mu}(u))$ is almost isomorphic to $L(\boldsymbol{a}, \boldsymbol{b})$.*

Proof. It follows from the proof of Theorem 4.11 that the vector v^+ is a highest $\ell_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-weight vector. Suppose the corresponding $\ell_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-weight is $\boldsymbol{\nu}(u)=(\nu_1(u),\nu_2(u))$. Then it follows from Proposition 2.9, Lemma 4.10, and Lemma 5.4, , that

$$\tilde\nu_1(u)=2\varepsilon_1 u\prod_{i=1}^k\frac{(u+s_1a_i)(u+s_1+s_1b_i)}{(u+s_1-s_1a_i)(u+s_1b_i)},\qquad \tilde\nu_2(u)=2\varepsilon_2 u\prod_{i=1}^k\frac{u-s_1b_i}{u+s_1b_i}.$$

Therefore, by (5.16) and (5.17), we have

$$\frac{\tilde\nu_1(u)}{\tilde\nu_2(u)}=\prod_{i=1}^k\frac{(u+s_1a_i)(u+s_1+s_1b_i)}{(u-s_1b_i)(u+s_1-s_1a_i)}=(-1)^{\deg P(u)}\frac{P(u)}{P(-u-s_1)}=\frac{\tilde\mu_1(u)}{\tilde\mu_2(u)}.$$

Thus, it suffices to prove that the $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-module $L(\boldsymbol{a},\boldsymbol{b})$ is irreducible.

We claim that any vector $\eta\in L(\boldsymbol{a},\boldsymbol{b})$ satisfying $b_{12}(u)\eta=0$ is proportional to v^+. We prove the claim by induction on k. The case $k=1$ is obvious by Lemma 5.4. Then we assume that $k\geqslant 2$. We write any such nonzero vector η in the following form,

$$\eta=\sum_{r=0}^{1}(e_{21})^r v_1^+\otimes\eta_r=v_1^+\otimes\eta_0+v_1^-\otimes\eta_1,$$

where $\eta_0,\eta_1\in L(a_2,b_2)\otimes\cdots\otimes L(a_k,b_k)$. We first prove that $\eta_1=0$. Suppose $\eta_1\neq 0$. Then it follows from Proposition 3.5 that

$$\begin{aligned}\Delta(b_{12}(u))=\,&t_{11}(u)t'_{12}(-u)\otimes b_{11}(u)+t_{11}(u)t'_{22}(-u)\otimes b_{12}(u)\\&-t_{12}(u)t'_{12}(-u)\otimes b_{21}(u)+t_{12}(u)t'_{22}(-u)\otimes b_{22}(u).\end{aligned}$$

Applying $b_{12}(u)$ to η using this coproduct, it follows from $b_{12}(u)\eta=0$ that

$$\big(t_{11}(u)t'_{22}(-u)\otimes b_{12}(u)\big)(v_1^-\otimes\eta_1)=0$$

by taking the coefficient of v_1^-. Therefore, we have

$$-\frac{(u-s_1+s_1a_1)(u-s_1a_1)}{(u+s_1-s_1a_1)(u+s_1b_1)}v_1^-\otimes b_{12}(u)\eta_1=0,$$

which implies that $b_{12}(u)\eta_1=0$. By induction hypothesis, the vector η_1 must be proportional to $v_2^+\otimes\cdots\otimes v_k^+$. Then taking the coefficient of v^+ in $b_{12}(u)\eta=0$, we have

$$0=\frac{u+s_1a_1}{u+s_1b_1}v_1^+\otimes b_{12}(u)\eta_0+t_{12}(u)t'_{22}(-u)v_1^-\otimes b_{22}(u)\eta_1+t_{11}(u)t'_{12}(-u)v_1^-\otimes b_{11}(u)\eta_1.$$

Note that by (4.15) we have $\tilde b_{11}(u)=(2u+s_1)b_{11}(u)-s_1b_{22}(u)$ and $\tilde b_{22}(u)=2ub_{22}(u)$. We deduce from the above equation and Lemma 4.10 that

$$\begin{aligned}0=\,&\frac{u+s_1a_1}{u+s_1b_1}v_1^+\otimes b_{12}(u)\eta_0+\frac{2\varepsilon_2s_1u(a_1+b_1)}{(u+s_1b_1)(2u+s_1)}\prod_{i=2}^k\frac{u-s_1b_i}{u+s_1b_i}v_1^+\otimes\eta_1\\&+\frac{2\varepsilon_1s_1u(a_1+b_1)(u+s_1a_1)}{(u+s_1-s_1a_1)(u+s_1b_1)(2u+s_1)}\prod_{i=2}^k\frac{(u+s_1a_i)(u+s_1+s_1b_i)}{(u+s_1-s_1a_i)(u+s_1b_i)}v_1^+\otimes\eta_1.\end{aligned}$$

Multiplying both sides by $(2u+s_1)\prod_{i=1}^k\big((u+s_1-s_1a_i)(u+s_1b_i)\big)$, we have

$$\begin{aligned}0=\,&(2u+s_1)(u+s_1a_1)(u+s_1-s_1a_1)v_1^+\otimes\prod_{i=2}^k\big((u+s_1-s_1a_i)(u+s_1b_i)\big)b_{12}(u)\eta_0\\&+2\varepsilon_2s_1u(a_1+b_1)(u+s_1-s_1a_1)\prod_{i=2}^k\big((u+s_1-s_1a_i)(u-s_1b_i)\big)v_1^+\otimes\eta_1\end{aligned}$$

$$+\,2\varepsilon_1 s_1 u(\mathscr{a}_1+\mathscr{b}_1)(u+s_1\mathscr{a}_1)\prod_{i=2}^{k}\big((u+s_1\mathscr{a}_i)(u+s_1+s_1\mathscr{b}_i)\big)v_1^+\otimes\eta_1.$$

Due to Lemma 5.4, the operator $\prod_{i=2}^{k}\big((u+s_1-s_1\mathscr{a}_i)(u+s_1\mathscr{b}_i)\big)b_{12}(u)$ acts on η_0 polynomially in u.

(1) If $\mathscr{a}_1\neq 0$, then setting $u=-s_1\mathscr{a}_1$, we obtain

$$2s_1\mathscr{a}_1(\mathscr{a}_1+\mathscr{b}_1)(2\mathscr{a}_1-1)\prod_{i=2}^{k}(\mathscr{a}_1+\mathscr{a}_i-1)(\mathscr{a}_1+\mathscr{b}_i)v_1^+\otimes\eta_1=0.$$

Thus, by (5.18), we conclude that $\eta_1=0$.

(2) If $\mathscr{a}_1=0$, then setting $u=s_1\mathscr{a}_1-s_1$, we get

$$2s_1(\mathscr{a}_1-1)(\mathscr{a}_1+\mathscr{b}_1)(2\mathscr{a}_1-1)\prod_{i=2}^{k}(\mathscr{a}_1+\mathscr{a}_i-1)(\mathscr{a}_1+\mathscr{b}_i)v_1^+\otimes\eta_1=0.$$

Again by (5.18), we conclude that $\eta_1=0$.

Therefore, we must have $b_{12}(u)\eta_0=0$ which again by induction hypothesis that η_0 is proportional to $v_2\otimes\cdots\otimes v_k^+$. Thus the claim is proved.

Suppose now that M is a submodule of $L(\boldsymbol{\mathscr{a}},\boldsymbol{\mathscr{b}})$. Then M must contain a nonzero vector η such that $b_{12}(u)\eta=0$, see Lemma 4.5. The above argument thus shows that M contains the vector v^+. It remains to prove the cyclic span $K=\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}v^+$ coincides with $L(\boldsymbol{\mathscr{a}},\boldsymbol{\mathscr{b}})$. By Lemma 5.6, the dual $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-module $L(\boldsymbol{\mathscr{a}},\boldsymbol{\mathscr{b}})^*$ is almost isomorphic to the restriction of the $\mathcal{Y}_{\boldsymbol{s}}$-module

$$L(\mathscr{b}_1+1,\mathscr{a}_1-1)\otimes\cdots\otimes L(\mathscr{b}_k+1,\mathscr{a}_k-1).$$

Moreover, the highest $\ell_{\boldsymbol{s},\boldsymbol{\varepsilon}}$ vector ζ_i^* of the module $L(\mathscr{b}_i+1,\mathscr{a}_i-1)\simeq L(\mathscr{a}_i,\mathscr{b}_i)^*$ can be identified with the elements of $L(\mathscr{a}_i,\mathscr{b}_i)^*$ such that $\zeta_i^*(v_i^+)=1$ and $\zeta_i^*(v_i^-)=0$. Now, if the submodule K of $L(\boldsymbol{\mathscr{a}},\boldsymbol{\mathscr{b}})$ is proper, then its annihilator

$$\operatorname{Ann}K:=\{\omega\in L(\boldsymbol{\mathscr{a}},\boldsymbol{\mathscr{b}})^*\mid \omega(\eta)=0\quad\text{for all}\quad \eta\in K\}$$

is a nonzero submodule of $L(\boldsymbol{\mathscr{a}},\boldsymbol{\mathscr{b}})^*$ which does not contain the vector $\zeta_1^*\otimes\cdots\otimes\zeta_k^*$. However, this contradicts the claim proved in the first part of the proof because the strategy still works for the module $L(\mathscr{b}_1+1,\mathscr{a}_1-1)\otimes\cdots\otimes L(\mathscr{b}_k+1,\mathscr{a}_k-1)$ with the previous assumptions on the complex numbers $\mathscr{a}_i,\mathscr{b}_i$. In this case, instead of using $2\mathscr{a}_1-1\neq 0$, we need the condition $2\mathscr{b}_1+1\neq 0$. This is true if $k\geqslant 2$ as the only possibility for $\mathscr{b}_i=-\frac{1}{2}$ is when $i=k$. As for the initial case $k=1$, it can be checked by a direct computation. ☐

Corollary 5.8. *Suppose $\boldsymbol{s}=(s_1,s_2)$ and $\boldsymbol{\varepsilon}=(\varepsilon_1,\varepsilon_2)$ are such that $s_1\neq s_2$ and $\varepsilon_1=\varepsilon_2$.*

(1) *If $\boldsymbol{\mu}(u)$ satisfies (5.16), where $P(u)$ and $P(-u-s_1)$ are relatively prime, then $\dim V(\boldsymbol{\mu}(u))=2^k$, where $k=\big\lfloor\frac{\deg P(u)+1}{2}\big\rfloor$.*

(2) *Given $k\in\mathbb{Z}_{>0}$, let $\mathscr{a}_i,\mathscr{b}_i$, $1\leqslant i\leqslant k$, be arbitrary complex numbers such that $\mathscr{a}_i+\mathscr{b}_i\neq 0$ and set*

$$P(u)=\prod_{i=1}^{k}\big((u+s_1\mathscr{a}_i)(u+s_1+s_1\mathscr{b}_i)\big).$$

Then the $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-module obtained by the restriction of the $\mathcal{Y}_{\boldsymbol{s}}$-module

$$L(\boldsymbol{\mathscr{a}},\boldsymbol{\mathscr{b}})=L(\mathscr{a}_1,\mathscr{b}_1)\otimes\cdots\otimes L(\mathscr{a}_k,\mathscr{b}_k)$$

is irreducible if and only if the greatest common divisor of $P(u)$ and $P(-u-s_1)$ (over $\mathbb{C}$) is of degree at most 1[2].

[2] If the greatest common divisor is nontrivial, then it has to be $u+\frac{s_1}{2}$.

5.5. **The case $\varepsilon_1 \neq \varepsilon_2$.** Now we assume that $\varepsilon_1 \neq \varepsilon_2$. In this case, it is slightly different from the previous case $\varepsilon_1 = \varepsilon_2$ as there is nontrivial one-dimensional module. We shall give detail for this part as well.

Suppose $V(\boldsymbol{\mu}(u))$ is finite dimensional, then by Proposition 5.3 we have

P-def2
$$\frac{\tilde{\mu}_1(u)}{\tilde{\mu}_2(u)} = (-1)^{\deg P(u)+1} \frac{P(u)}{P(-u-s_1)}. \tag{5.19}$$

We assume further that $P(u)$ and $P(-u-s_1)$ are relatively prime. Otherwise, we may cancel common factors and obtain a polynomial of smaller degree. Then $P(-\frac{s_1}{2}) \neq 0$. Suppose

$$P(u) = (u + s_1\mathscr{a}_1)(u + s_1\mathscr{a}_2)\cdots(u + s_1\mathscr{a}_l)$$

where $l = \deg P(u)$ and $\mathscr{a}_i \in \mathbb{C}$ for $1 \leqslant i \leqslant l$. We have $\mathscr{a}_i \neq \frac{1}{2}$.

Set $k = \lfloor \frac{l}{2} \rfloor$. Introduce k pair of complex numbers $(\mathscr{a}_i, \mathscr{b}_i)$, where $\mathscr{a}_i$ are defined as above while $\mathscr{b}_i$ are defined as follows,

- if l is odd, then $\mathscr{b}_i = \mathscr{a}_{i+k} - 1$ for $1 \leqslant i \leqslant k$;
- if l is even, then $\mathscr{b}_i = \mathscr{a}_{i+k} - 1$ for $1 \leqslant i < k$ and $\mathscr{b}_k = -\frac{1}{2}$.

We also set $\gamma = \varepsilon_1 s_1(a_l - 1)$. Then we have

P-factor2
$$\frac{P(u)}{P(-u-s_1)} = (-1)^{\deg P(u)+1} \frac{\varepsilon_1 u + s_1\varepsilon_1 + \gamma}{\varepsilon_2 u + \gamma} \prod_{i=1}^{k} \frac{(u + s_1\mathscr{a}_i)(u + s_1 + s_1\mathscr{b}_i)}{(u - s_1\mathscr{b}_i)(u + s_1 - s_1\mathscr{a}_i)}. \tag{5.20}$$

The only possible cancellation is when l is even, then

$$u + s_1 + s_1\mathscr{b}_k = u - s_1\mathscr{b}_k.$$

Note that we have

notequalto2
$$\mathscr{a}_i + \mathscr{b}_i \neq 0, \quad \mathscr{a}_i + \mathscr{b}_j \neq 0, \quad \mathscr{a}_i + \mathscr{a}_j \neq 1, \quad \mathscr{b}_i + \mathscr{b}_j \neq 1 \tag{5.21}$$

for all $1 \leqslant i \neq j \leqslant k$.

We consider the tensor product of evaluation $\mathscr{Y}_{\boldsymbol{s}}$-modules,

$$L(\boldsymbol{\mathscr{a}}, \boldsymbol{\mathscr{b}}) = L(\mathscr{a}_1, \mathscr{b}_1) \otimes \cdots \otimes L(\mathscr{a}_k, \mathscr{b}_k)$$

and the tensor product

$$V_\gamma(\boldsymbol{\mathscr{a}}, \boldsymbol{\mathscr{b}}) = L(\boldsymbol{\mathscr{a}}, \boldsymbol{\mathscr{b}}) \otimes \mathbb{C}_\gamma$$

Then $V_\gamma(\boldsymbol{\mathscr{a}}, \boldsymbol{\mathscr{b}})$ is a $\mathscr{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-module.

For each $1 \leqslant i \leqslant k$, $L(\mathscr{a}_i, \mathscr{b}_i)$ is two dimensional. We set v_i^+ to be one of its nonzero singular vector and $v_i^- = e_{21}v_i^+$. Moreover, we assume that v_i^+ are even. Suppose $\mathbb{C}_\gamma$ is spanned by v_0. We also set $v^+ = v_1^+ \otimes \cdots \otimes v_k^+ \otimes v_0$.

Proposition 5.9. *If $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2)$ satisfies $\varepsilon_1 \neq \varepsilon_2$, then the $\mathscr{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-module $V_\gamma(\boldsymbol{\mathscr{a}}, \boldsymbol{\mathscr{b}})$ is irreducible. Moreover, the finite-dimensional irreducible $\mathscr{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-module $V(\boldsymbol{\mu}(u))$ is almost isomorphic to $V_\gamma(\boldsymbol{\mathscr{a}}, \boldsymbol{\mathscr{b}})$.*

Proof. It follows from the proof of Theorem 4.11 that the vector v^+ is a highest $\ell_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-weight vector. Suppose the corresponding $\ell_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-weight is $\boldsymbol{\nu}(u) = (\nu_1(u), \nu_2(u))$. Then it follows from Proposition 2.9, Proposition 4.13, and Lemma 5.4, that

$$\tilde{\nu}_1(u) = 2(\varepsilon_1 u + s_1\varepsilon_1 + \gamma) \prod_{i=1}^{k} \frac{(u + s_1\mathscr{a}_i)(u + s_1 + s_1\mathscr{b}_i)}{(u + s_1 - s_1\mathscr{a}_i)(u + s_1\mathscr{b}_i)}, \quad \tilde{\nu}_2(u) = 2(\varepsilon_2 u + \gamma) \prod_{i=1}^{k} \frac{u - s_1\mathscr{b}_i}{u + s_1\mathscr{b}_i}.$$

Therefore, by (5.16) and (5.17), we have

$$\begin{aligned} \frac{\tilde{\nu}_1(u)}{\tilde{\nu}_2(u)} &= \frac{\varepsilon_1 u + s_1\varepsilon_1 + \gamma}{\varepsilon_2 u + \gamma} \prod_{i=1}^{k} \frac{(u + s_1\mathscr{a}_i)(u + s_1 + s_1\mathscr{b}_i)}{(u - s_1\mathscr{b}_i)(u + s_1 - s_1\mathscr{a}_i)} \\ &= (-1)^{\deg P(u)+1} \frac{P(u)}{P(-u-s_1)} = \frac{\tilde{\mu}_1(u)}{\tilde{\mu}_2(u)}. \end{aligned}$$

Thus, it suffices to prove that the $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-module $V_\gamma(\boldsymbol{a},\boldsymbol{b})$ is irreducible.

We claim that any vector $\eta \in V_\gamma(\boldsymbol{a},\boldsymbol{b})$ satisfying $b_{12}(u)\eta = 0$ is proportional to v^+. We prove the claim by induction on k. The case $k = 1$ is prove by a direct computation using Lemma 5.4. Then we assume that $k \geqslant 2$. We write any such nonzero vector η in the following form,

$$\eta = \sum_{r=0}^{1} (e_{21})^r v_1^+ \otimes \eta_r = v_1^+ \otimes \eta_0 + v_1^- \otimes \eta_1,$$

where $\eta_0, \eta_1 \in L(a_2, b_2) \otimes \cdots \otimes L(a_k, b_k) \otimes \mathbb{C}_\gamma$. Similar to the proof of Proposition 5.7, the vector η_1 must be proportional to $v_2^+ \otimes \cdots \otimes v_k^+$. Then taking the coefficient of v^+ in $b_{12}(u)\eta = 0$, we have

$$0 = \frac{u + s_1 a_1}{u + s_1 b_1} v_1^+ \otimes b_{12}(u)\eta_0 + t_{12}(u)t'_{22}(-u)v_1^- \otimes b_{22}(u)\eta_1 + t_{11}(u)t'_{12}(-u)v_1^- \otimes b_{11}(u)\eta_1.$$

Note that by (4.15) we have $\tilde{b}_{11}(u) = (2u + s_1)b_{11}(u) - s_1 b_{22}(u)$ and $\tilde{b}_{22}(u) = 2u b_{22}(u)$. We deduce from the above equation and Lemma 4.10 that

$$\begin{aligned} 0 = {} & \frac{u + s_1 a_1}{u + s_1 b_1} v_1^+ \otimes b_{12}(u)\eta_0 + \frac{2s_1 u(a_1 + b_1)(\varepsilon_2 u + \gamma)}{(u + s_1 b_1)(2u + s_1)(u - \gamma)} \prod_{i=2}^{k} \frac{u - s_1 b_i}{u + s_1 b_i} v_1^+ \otimes \eta_1 \\ & + \frac{2s_1 u(a_1 + b_1)(u + s_1 a_1)(\varepsilon_1 u + s_1\varepsilon_1 + \gamma)}{(u + s_1 - s_1 a_1)(u + s_1 b_1)(2u + s_1)(u - \gamma)} \prod_{i=2}^{k} \frac{(u + s_1 a_i)(u + s_1 + s_1 b_i)}{(u + s_1 - s_1 a_i)(u + s_1 b_i)} v_1^+ \otimes \eta_1. \end{aligned}$$

Multiplying both sides by $(2u + s_1)(u - \gamma)\prod_{i=1}^{k}\big((u + s_1 - s_1 a_i)(u + s_1 b_i)\big)$, we have

$$\begin{aligned} 0 = {} & (u - \gamma)(2u + s_1)(u + s_1 a_1)(u + s_1 - s_1 a_1)v_1^+ \otimes \prod_{i=2}^{k}\big((u + s_1 - s_1 a_i)(u + s_1 b_i)\big) b_{12}(u)\eta_0 \\ & + 2s_1 u(\varepsilon_2 u + \gamma)(a_1 + b_1)(u + s_1 - s_1 a_1)\prod_{i=2}^{k}\big((u + s_1 - s_1 a_i)(u - s_1 b_i)\big) v_1^+ \otimes \eta_1 \\ & + 2s_1 u(\varepsilon_1 u + s_1\varepsilon_1 + \gamma)(a_1 + b_1)(u + s_1 a_1)\prod_{i=2}^{k}\big((u + s_1 a_i)(u + s_1 + s_1 b_i)\big) v_1^+ \otimes \eta_1. \end{aligned}$$

The rest of the proof is parallel to that of Proposition 5.7. Again, we need the condition that the only possible cancellation in the right hand side of (5.20) is when l is even, then $u + s_1 + s_1 b_k = u - s_1 b_k$. ☐

Corollary 5.10. *Suppose $\boldsymbol{s} = (s_1, s_2)$ and $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2)$ are such that $s_1 \neq s_2$ and $\varepsilon_1 \neq \varepsilon_2$.*

(1) *If $\boldsymbol{\mu}(u)$ satisfies (5.19), where $P(u)$ and $P(-u - s_1)$ are relatively prime, then $\dim V(\boldsymbol{\mu}(u)) = 2^k$, where $k = \lfloor \frac{\deg P(u)}{2} \rfloor$.*

(2) *Given $k \in \mathbb{Z}_{>0}$, let γ, a_i, b_i, $1 \leqslant i \leqslant k$, be arbitrary complex numbers such that $a_i + b_i \neq 0$ and set*

$$P(u) = (u + s_1 + \varepsilon_1\gamma)\prod_{i=1}^{k}\big((u + s_1 a_i)(u + s_1 + s_1 b_i)\big).$$

Then the $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-module

$$V_\gamma(\boldsymbol{a},\boldsymbol{b}) = L(a_1, b_1) \otimes \cdots \otimes L(a_k, b_k) \otimes \mathbb{C}_\gamma$$

is irreducible if and only if the greatest common divisor of $P(u)$ and $P(-u - s_1)$ (over $\mathbb{C}$) is of degree at most 1.

6. Classification in higher ranks

sec:classification

6.1. Sufficient conditions.

We have the following sufficient condition for $V(\boldsymbol{\mu}(u))$ being finite-dimensional for arbitrary $\boldsymbol{s}$ and arbitrary $\boldsymbol{\varepsilon}$.

thm:suff

Theorem 6.1. *Suppose the highest $\ell_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-weight $\boldsymbol{\mu}(u)$ satisfies*

eq:in-proof-01

$$\frac{\tilde{\mu}_i(u)}{\tilde{\mu}_{i+1}(u)}=\frac{(2\varepsilon_i u-\varepsilon_i\rho_{i+1}+\varpi_{i+1}+2\gamma)\lambda_i(u)\lambda_{i+1}(-u+\rho_{i+1})}{(2\varepsilon_{i+1}u-\varepsilon_{i+1}\rho_{i+2}+\varpi_{i+2}+2\gamma)\lambda_{i+1}(u)\lambda_i(-u+\rho_{i+1})},\qquad 1\leqslant i<\varkappa, \tag{6.1}$$

where $\gamma\in\mathbb{C}$ and $\boldsymbol{\lambda}(u)=(\lambda_i(u))_{1\leqslant i\leqslant\varkappa}$ is an $\ell_{\boldsymbol{s}}$-weight such that the $\mathscr{Y}_{\boldsymbol{s}}$-module $L(\boldsymbol{\lambda}(u))$ is finite-dimensional, then $V(\boldsymbol{\mu}(u))$ is finite-dimensional.

Proof. Let $\boldsymbol{\lambda}(u)=(\lambda_i(u))_{1\leqslant i\leqslant\varkappa}$ be an $\ell_{\boldsymbol{s}}$-weight such that the $\mathscr{Y}_{\boldsymbol{s}}$-module $L(\boldsymbol{\lambda}(u))$ is finite-dimensional. Suppose its highest $\ell_{\boldsymbol{s}}$-weight vector is ξ. Consider $\mathscr{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$ as a subalgebra as in Proposition 3.3. Let $\mathbb{C}_\gamma$ be the one-dimensional $\mathscr{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-module spanned by η_γ as in Example 4.2. Then $L(\boldsymbol{\lambda}(u))\otimes\mathbb{C}_\gamma$ is a $\mathscr{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-module, see Example 4.14.

It follows from Proposition 4.13 and Example 4.14 that ξ is a highest $\ell_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-weight with $\ell_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-weight $\boldsymbol{\zeta}(u)$ such that

eq:in-proof-02

$$\frac{\tilde{\zeta}_i(u)}{\tilde{\zeta}_{i+1}(u)}=\frac{(2\varepsilon_i u-\varepsilon_i\rho_{i+1}+\varpi_{i+1}+2\gamma)\lambda_i(u)\lambda_{i+1}(-u+\rho_{i+1})}{(2\varepsilon_{i+1}u-\varepsilon_{i+1}\rho_{i+2}+\varpi_{i+2}+2\gamma)\lambda_{i+1}(u)\lambda_i(-u+\rho_{i+1})},\qquad 1\leqslant i<\varkappa. \tag{6.2}$$

Let $\mathscr{M}:=\mathscr{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}(\xi\otimes\eta)$. Then $\mathscr{M}$ is a highest $\ell_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-weight with $\ell_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-weight $\boldsymbol{\zeta}(u)$. Moreover $\mathscr{M}$ is finite-dimensional as a subspace of $L(\boldsymbol{\lambda}(u))$.

Note that the series $f(u)=\mu_\varkappa(u)/\zeta_\varkappa(u)\in 1+u^{-1}\mathbb{C}[[u^{-1}]]$ satisfies $f(u)f(-u)=1$, see (4.22). Denote by $\mathscr{M}^{f(u)}$ the $\mathscr{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-module obtained by pulling back $\mathscr{M}$ through the automorphism defined by $b_{ij}(u)\mapsto f(u)b_{ij}(u)$, for $1\leqslant i,j,\leqslant\varkappa$. Comparing (6.1) and (6.2), we see that $\mathscr{M}^{f(u)}$ is a highest $\ell_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-weight with $\ell_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-weight $\boldsymbol{\mu}(u)$. Therefore $V(\boldsymbol{\mu}(u))$ is finite-dimensional. ☐

It is very nature to expect that this is also the necessary condition for $V(\boldsymbol{\mu}(u))$ being finite-dimensional.

conj:main

Conjecture 6.2. *If the irreducible $\mathscr{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-module $V(\boldsymbol{\mu}(u))$ is finite-dimensional, then there exist $\gamma\in\mathbb{C}$ and an $\ell_{\boldsymbol{s}}$-weight $\boldsymbol{\lambda}(u)=(\lambda_i(u))_{1\leqslant i\leqslant\varkappa}$ such that*

(1) *the equations (6.1) hold, and*
(2) *the $\mathscr{Y}_{\boldsymbol{s}}$-module $L(\boldsymbol{\lambda}(u))$ is finite-dimensional.*

We call $\boldsymbol{\varepsilon}$ *simple* if there exists at most one $1\leqslant i<\varkappa$ such that $\varepsilon_i\neq\varepsilon_{i+1}$. Conjecture 6.2 is proved in [MR02] for the case that $\boldsymbol{\varepsilon}$ is simple and $n=0$ (non-super case). Our main results in this section are to show the conjecture for (1) the case when $n=0,1$ and $\boldsymbol{\varepsilon}$ *is arbitrary*; and (2) the case when $\boldsymbol{s}$ is the standard parity sequence and $\boldsymbol{\varepsilon}$ is simple. The main obstacle for the super case is that when $\boldsymbol{s}$ is not the standard parity sequence, an explicit criterion for the $\mathscr{Y}_{\boldsymbol{s}}$-module $L(\boldsymbol{\lambda}(u))$ being finite-dimensional is not available, though a recursive criterion can be deduced from [Mol22, Lu22].

sec:reduct-lem

6.2. Reduction lemmas.

In this subsection, we prepare reduction lemmas which allows us to construct modules of twisted Yangians of lower ranks from modules of twisted Yangians of higher ranks.

For given $\boldsymbol{s}\in S_{m|n}$ and $\boldsymbol{\varepsilon}$, define

$$\begin{aligned}\overline{\boldsymbol{s}}&=(s_2,s_3,\cdots,s_\varkappa), &\qquad \overline{\boldsymbol{\varepsilon}}&=(\varepsilon_2,\varepsilon_3,\cdots,\varepsilon_\varkappa),\\ \underline{\boldsymbol{s}}&=(s_1,s_2,\cdots,s_{\varkappa-1}), &\qquad \underline{\boldsymbol{\varepsilon}}&=(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_{\varkappa-1}).\end{aligned}\tag{6.3}$$

Then we have twisted super Yangians $\mathscr{B}_{\overline{\boldsymbol{s}},\overline{\boldsymbol{\varepsilon}}}$ and $\mathscr{B}_{\underline{\boldsymbol{s}},\underline{\boldsymbol{\varepsilon}}}$. To distinguish the underlying generating series, we rewrite the series $b_{ij}(u)$ as $b^\circ_{ij}(u)$ in $\mathscr{B}_{\overline{\boldsymbol{s}},\overline{\boldsymbol{\varepsilon}}}$ or $\mathscr{B}_{\underline{\boldsymbol{s}},\underline{\boldsymbol{\varepsilon}}}$.

Let V be a representation of the twisted super Yangian $\mathscr{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$. Define

Vover

$$\overline{V}=\{v\in V\mid b_{1i}(u)v=0,\ 1<i\leqslant\varkappa\}.\tag{6.4}$$

lem:embedding-over

Lemma 6.3. *The map* $\mathcal{B}_{\overline{s},\overline{\varepsilon}} \to \mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$ *defined by* $b^{\circ}_{ij}(u) \to b_{i+1,j+1}(u)$, $1 \leqslant i,j < \varkappa$, *induces a representation of* $\mathcal{B}_{\overline{s},\overline{\varepsilon}}$ *on* $\overline{V}$.

Proof. This follows immediately from (3.3). ☐

Similarly, define

$$\underline{V} = \{v \in V \mid b_{i\varkappa}(u)v = 0,\ 1 \leqslant i < \varkappa\}. \tag{6.5}$$

Vund

lem:embedding-under

Lemma 6.4 ([BR09, Theorem 3.1])**.** *The map* $\mathcal{B}_{\underline{s},\underline{\varepsilon}} \to \mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$ *defined by*

$$b^{\circ}_{ij}(u) \to b_{ij}\Big(u + \frac{s_\varkappa}{2}\Big) + \delta_{ij}\frac{s_\varkappa}{2u} b_{\varkappa\varkappa}\Big(u + \frac{s_\varkappa}{2}\Big)$$

induces a representation of $\mathcal{B}_{\underline{s},\underline{\varepsilon}}$ *on* $\underline{V}$. ☐

Fix $1 \leqslant a < \varkappa$. Let

$$\boldsymbol{s}^{\star} = (s_a, s_{a+1}), \qquad \boldsymbol{s}[a] = (s_1, \cdots, s_a, s_{a+1}),$$
$$\boldsymbol{\varepsilon}^{\star} = (\varepsilon_a, \varepsilon_{a+1}), \qquad \boldsymbol{\varepsilon}[a] = (\varepsilon_1, \cdots, \varepsilon_a, \varepsilon_{a+1}).$$

For a $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-module V, define

$$\begin{aligned} V^{\star} = \{\eta \in V \,|\, & b_{ij}(u)\eta = 0,\ 1 \leqslant i < a,\ i < j \leqslant \varkappa, \\ & b_{kl}(u)\eta = 0,\ a+1 < l \leqslant \varkappa,\ 1 \leqslant k < l\}. \end{aligned} \tag{6.6}$$

vstar

B2reduction

Lemma 6.5. *The map*

$$\mathcal{B}_{\boldsymbol{s}^\star,\boldsymbol{\varepsilon}^\star} \to \mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}, \quad b^{\star}_{ij}(u) \mapsto b_{a+i-1,a+j-1}\Big(u + \frac{\rho_{a+2}}{2}\Big) + \frac{\delta_{ij}}{2u}\sum_{k=a+2}^{\varkappa} s_k b_{kk}\Big(u + \frac{\rho_{a+2}}{2}\Big)$$

induces a representation of $\mathcal{B}_{\boldsymbol{s}^\star,\boldsymbol{\varepsilon}^\star}$ *on* $V^\star$. *Moreover, under this map, we have*

$$\tilde{b}^{\star}_{ii}(u) \mapsto \tilde{b}_{a+i-1,a+i-1}\Big(u + \frac{\rho_{a+2}}{2}\Big), \qquad i = 1, 2.$$

Proof. The first statement follows from repeatedly applying Lemma 6.3 and Lemma 6.4. The second one is obvious. ☐

Note that if V is finite-dimensional, then, by Lemma 4.5, none of $\overline{V}$, $\underline{V}$, and $V^\star$ is trivial.

6.3. **Classification 1.** Let $\sigma \in \mathfrak{S}_\varkappa$. Given $\boldsymbol{s}$ and $\boldsymbol{\varepsilon}$, define

$$\boldsymbol{s}^{\sigma} = (s_{\sigma^{-1}(1)}, \cdots, s_{\sigma^{-1}(\varkappa)}), \quad \boldsymbol{\varepsilon}^{\sigma} = (\varepsilon_{\sigma^{-1}(1)}, \cdots, \varepsilon_{\sigma^{-1}(\varkappa)}).$$

Then we have the following natural isomorphisms, which by abuse of notation we denote by σ again,

$$\sigma : \mathcal{Y}_{\boldsymbol{s}} \to \mathcal{Y}_{\boldsymbol{s}^\sigma}, \qquad t^{\boldsymbol{s}}_{ij}(u) \mapsto t^{\boldsymbol{s}^\sigma}_{\sigma(i)\sigma(j)}(u), \tag{6.7}$$

sigmaY

and

$$\sigma : \mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}} \to \mathcal{B}_{\boldsymbol{s}^\sigma,\boldsymbol{\varepsilon}^\sigma}, \qquad b^{\boldsymbol{s}}_{ij}(u) \mapsto b^{\boldsymbol{s}^\sigma}_{\sigma(i)\sigma(j)}(u). \tag{6.8}$$

sigmaB

Note that the latter one is the same as the one obtained by the restriction of the former.

Fix $1 \leqslant a < \varkappa$, $\boldsymbol{s}$, and $\boldsymbol{\varepsilon}$, we shall denote

$$\tilde{\boldsymbol{s}} := \boldsymbol{s}^{\sigma} = (s_1, \cdots, s_{a+1}, s_a, \cdots, s_\varkappa),$$
$$\tilde{\boldsymbol{\varepsilon}} := \boldsymbol{\varepsilon}^{\sigma} = (\varepsilon_1, \cdots, \varepsilon_{a+1}, \varepsilon_a, \cdots, \varepsilon_\varkappa),$$

We shall identify the superalgebra $\mathcal{Y}_{\tilde{\boldsymbol{s}}}$ with $\mathcal{Y}_{\boldsymbol{s}}$ via the isomorphism (6.7), $\mathcal{B}_{\tilde{\boldsymbol{s}},\tilde{\boldsymbol{\varepsilon}}}$ with $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$ via the isomorphism (6.8). When the underlying parity sequence and the sequence $\boldsymbol{\varepsilon}$ are omitted, we implicitly assume that we pick our fixed choice of $\boldsymbol{s}$ and $\boldsymbol{\varepsilon}$. Then we can rephrase the conditions for a vector being a highest $\ell_{\tilde{\boldsymbol{s}}}$-weight vector ζ of $\ell_{\tilde{\boldsymbol{s}}}$-weight $\boldsymbol{\nu}(u)$ as follows,

$$t_{ii}(u)\zeta = \nu_i(u)\zeta, \quad t_{aa}(u)\zeta = \nu_{a+1}(u)\zeta, \quad t_{a+1,a+1}(u)\zeta = \nu_a(u)\zeta,$$

$$t_{i,i+1}(u)\zeta = t_{a-1,a+1}(u)\zeta = t_{a+1,a}(u)\zeta = t_{a,a+2}(u)\zeta = 0, \quad i \neq a-1, a, a+1.$$

Recall that $L(\boldsymbol{\lambda}(u))$ is the irreducible $\mathscr{Y}_{\boldsymbol{s}}$-module of highest $\ell_{\boldsymbol{s}}$-weight $\boldsymbol{\lambda}(u)$. Suppose $L(\boldsymbol{\lambda}(u))$ is finite-dimensional and $s_a \neq s_{a+1}$, then it follows from [Zha95] that $\lambda_a(u)/\lambda_{a+1}(u)$ is a series in u^{-1} as a rational function expanded at $u = \infty$. Let

pqdef
$$\frac{\lambda_a(u)}{\lambda_{a+1}(u)} = \frac{\mathscr{p}(u)}{\mathscr{q}(u)}, \tag{6.9}$$

where $\mathscr{p}(u)$ and $\mathscr{q}(u)$ are relatively prime monic polynomials in u of the same degree. Set

mcqdeg
$$\deg \mathscr{q}(u) = k. \tag{6.10}$$

We also need the following

lem:ell-weight-ref

Lemma 6.6. *Suppose $L(\boldsymbol{\lambda}(u))$ is finite-dimensional, then $L(\boldsymbol{\lambda}(u))$ contains a unique highest $\ell_{\tilde{\boldsymbol{s}}}$-weight vector (up to proportionality) of $\boldsymbol{\nu}(u)$, where $\boldsymbol{\nu}(u)$ is given by the following rules,*

(1) *if $s_a = s_{a+1}$, then $\boldsymbol{\nu}(u) = \boldsymbol{\lambda}(u)$;*
(2) *if $s_a \neq s_{a+1}$, then $\nu_i(u) = \lambda_i(u)$ for $i \neq a, a+1$ and*

$$\nu_a(u) = \lambda_{a+1}(u)\frac{\mathscr{q}(u - s_a)}{\mathscr{q}(u)}, \quad \nu_{a+1}(u) = \lambda_a(u)\frac{\mathscr{p}(u - s_a)}{\mathscr{p}(u)}.$$

Proof. Case (1) is probably well known[3] and case (2) follows from the odd reflections of super Yangians, see [Mol22, Lu22]. $\square$

thm:reflection

Theorem 6.7. *If Conjecture 6.2 holds true for the case $\boldsymbol{\varepsilon}$, then it also holds true for the case $\boldsymbol{\varepsilon}^\sigma$ for any $\sigma \in \mathfrak{S}_\varkappa$.*

Proof. It suffices to prove it for the case $\sigma = \sigma_a = (a, a+1)$ for any fix $1 \leqslant a < \varkappa$. We shall use the notations introduced above. Since there is a single choice of $\boldsymbol{s}$, we shall drop the dependence on $\boldsymbol{s}$ for the notations. Also, to distinguish $\ell_{\boldsymbol{s}}$ and $\ell_{\tilde{\boldsymbol{s}}}$, we use the notation ℓ and $\tilde{\ell}$ instead, respectively[4].

Let $\boldsymbol{\mu}(u)$ be a highest $\ell_{\tilde{\boldsymbol{\varepsilon}}}$-weight. Suppose the irreducible $\mathcal{B}_{\tilde{\boldsymbol{\varepsilon}}}$-module $V(\boldsymbol{\mu}(u))$ is finite-dimensional. Since $\mathcal{B}_{\tilde{\boldsymbol{\varepsilon}}}$ and $\mathcal{B}_{\boldsymbol{\varepsilon}}$ are isomorphic, the $\mathcal{B}_{\tilde{\boldsymbol{\varepsilon}}}$-module $V(\boldsymbol{\mu}(u))$ is also a finite-dimensional irreducible $\mathcal{B}_{\boldsymbol{\varepsilon}}$-module. Suppose the highest $\ell_{\boldsymbol{\varepsilon}}$-weight of $V(\boldsymbol{\mu}(u))$ is $\boldsymbol{\nu}(u)$, that is the $\mathcal{B}_{\tilde{\boldsymbol{\varepsilon}}}$-module $V(\boldsymbol{\mu}(u))$ is almost isomorphic to the $\mathcal{B}_{\boldsymbol{\varepsilon}}$-module $V(\boldsymbol{\nu}(u))$ if we identify $\mathcal{B}_{\tilde{\boldsymbol{\varepsilon}}}$ with $\mathcal{B}_{\boldsymbol{\varepsilon}}$. By assumption, Conjecture 6.2 is true for the case $\boldsymbol{\varepsilon}$, therefore, there exists a highest ℓ-weight $\boldsymbol{\lambda}(u)$ and $\gamma \in \mathbb{C}$ such that (6.1) are satisfied and the $\mathscr{Y}$-module $L(\boldsymbol{\lambda}(u))$ is finite-dimensional too. Note that the choice of $\boldsymbol{\lambda}(u)$ may not be unique. We shall pick a particular $\boldsymbol{\lambda}(u)$ and sketch the proof (a complete proof will be added in a later version).

We pick $\boldsymbol{\lambda}(u)$ such that the finite-dimensional irreducible $\mathscr{Y}(\mathfrak{gl}_2)$-module $L(\lambda_a(u), \lambda_{a+1}(u))$ tensor with the one-dimensional module $\mathbb{C}_\gamma$ restricts to an irreducible $\mathcal{B}_2$-module, see Proposition 5.2. Then the highest $\tilde{\ell}$-weight vector corresponds to the lowest ℓ-weight vector in the $\mathscr{Y}(\mathfrak{gl}_2)$-module $L(\lambda_a(u), \lambda_{a+1}(u))$ (since such a vector is unique as it is "highest weight" in terms of the usual weight) and hence this vector corresponds to a highest $\ell_{\tilde{\varepsilon}}$-weight vector. However, the corresponding ℓ-weight does not change by Lemma 6.6. Hence to compute the $\ell_{\tilde{\varepsilon}}$-weight, one only needs to change $\boldsymbol{\varepsilon}$ to $\tilde{\boldsymbol{\varepsilon}}$. $\square$

[3]It essentially reduces to the case of $\mathscr{Y}(\mathfrak{gl}_2)$ where all finite-dimensional irreducible modules are almost isomorphic to tensor product of evaluation modules.

[4]Though $\boldsymbol{s} = \tilde{\boldsymbol{s}}$, the $\ell_{\boldsymbol{s}}$-weight and $\ell_{\tilde{\boldsymbol{s}}}$-weight have slightly different meaning.

6.4. **Classification 2.**

thm:main-super-class

Theorem 6.8. *Suppose $\boldsymbol{s}$ is the standard parity sequence and $\boldsymbol{\varepsilon}$ is simple. The $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$ module $V(\boldsymbol{\mu}(u))$ is finite-dimensional if and only if there exists $\gamma\in\mathbb{C}$ and for any $1\leqslant i<\varkappa$ such that*

(1) *if $s_i=s_{i+1}$, there exists a monic polynomial $P_i(u)$ satisfying $P_i(u)=P_i(-u+\rho_i)$,*

$$\frac{\tilde{\mu}_i(u)}{\tilde{\mu}_{i+1}(u)}=\frac{(2\varepsilon_i u-\varepsilon_i\rho_{i+1}+\varpi_{i+1}+\gamma)P_i(u+s_i)}{(2\varepsilon_{i+1}u-\varepsilon_{i+1}\rho_{i+2}+\varpi_{i+2}+\gamma)P_i(u)}. \tag{6.11}$$

eq:gamma-thm

Moreover, if $\varepsilon_i\neq\varepsilon_{i+1}$, then $P_i(u)$ is not divisible by $2\varepsilon_i u-\varepsilon_i\rho_{i+1}+\varpi_{i+1}+\gamma$.

(2) *if $s_i\neq s_{i+1}$, there exists a monic polynomials $P_i(u)$ satisfying*

$$\frac{\tilde{\mu}_i(u)}{\tilde{\mu}_{i+1}(u)}=\varepsilon_i\varepsilon_{i+1}(-1)^{\deg P_i}\frac{P_i(u)}{P_i(-u+\rho_{i+1})}.$$

Proof. By Theorem 6.1, it suffices to prove the "only if" part. Let $V=V(\boldsymbol{\mu}(u))$ and assume that V is finite-dimensional. We proceed by induction on n. For the base case $n=2$, it follows from Propositions 5.1 and 5.2. Then we assume that $n\geqslant 3$.

Recall the notation from §6.2 and set ξ to be the highest $\ell_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-weight vector. Consider the subspace $\overline{V}$ defined in (6.4), then $\overline{V}$ is a finite-dimensional $\mathcal{B}_{\overline{\boldsymbol{s}},\overline{\boldsymbol{\varepsilon}}}$-module by Lemma 6.3. Clearly, $\xi\in\overline{V}$ and ξ is a highest $\ell_{\overline{\boldsymbol{s}},\overline{\boldsymbol{\varepsilon}}}$-weight vector of the $\ell_{\overline{\boldsymbol{s}},\overline{\boldsymbol{\varepsilon}}}$-weight $\overline{\boldsymbol{\mu}}(u)=(\mu_2(u),\cdots,\mu_\varkappa(u))$. Thus the cyclic span $\mathcal{B}_{\overline{\boldsymbol{s}},\overline{\boldsymbol{\varepsilon}}}\xi$ is a finite-dimensional highest $\ell_{\overline{\boldsymbol{s}},\overline{\boldsymbol{\varepsilon}}}$-weight with highest $\ell_{\overline{\boldsymbol{s}},\overline{\boldsymbol{\varepsilon}}}$-weight $\overline{\boldsymbol{\mu}}(u)$. In particular, $V(\overline{\boldsymbol{\mu}}(u))$ is finite-dimensional. By induction hypothesis, we conclude that the conditions from the theorem are satisfied for the components of $\overline{\boldsymbol{\mu}}(u)$ (that is for $1<i<\varkappa$) for some $\gamma_1\in\mathbb{C}$.

Similarly, consider the subspace $\underline{V}$ defined in (6.5), then $\underline{V}$ is a finite-dimensional $\mathcal{B}_{\underline{\boldsymbol{s}},\underline{\boldsymbol{\varepsilon}}}$-module by Lemma 6.4. Clearly, $\xi\in\underline{V}$ and ξ is a highest $\ell_{\underline{\boldsymbol{s}},\underline{\boldsymbol{\varepsilon}}}$-weight vector with the $\ell_{\underline{\boldsymbol{s}},\underline{\boldsymbol{\varepsilon}}}$-weight

$$\underline{\boldsymbol{\mu}}^\circ(u)=\Big(\mu_1\Big(u+\frac{s_\varkappa}{2}\Big)+\frac{s_\varkappa}{2u}\mu_\varkappa\Big(u+\frac{s_\varkappa}{2}\Big),\cdots,\mu_{\varkappa-1}\Big(u+\frac{s_\varkappa}{2}\Big)+\frac{s_\varkappa}{2u}\mu_\varkappa\Big(u+\frac{s_\varkappa}{2}\Big)\Big).$$

Let $\tilde{\mu}_i^\circ(u)$ be the series associated to $\boldsymbol{\mu}^\circ(u)$ as defined in (4.24). Then it is clear that

$$\tilde{\mu}_i^\circ(u)=\tilde{\mu}_i\Big(u+\frac{s_\varkappa}{2}\Big).$$

By the same argument as in the previous paragraph, we conclude that the conditions from the theorem are satisfied for $1\leqslant i<\varkappa-1$ for some $\gamma_2\in\mathbb{C}$.

Now it suffices to show that we can choose $\gamma_1=\gamma_2$. Recall from (4.25) that if $\varepsilon_i=\varepsilon_{i+1}$, for $1\leqslant i<\varkappa$, then

$$2\varepsilon_i u-\varepsilon_i\rho_{i+1}+\varpi_{i+1}=2\varepsilon_{i+1}u-\varepsilon_{i+1}\rho_{i+2}+\varpi_{i+2}.$$

Hence the number γ only shows up in (6.11) when $\varepsilon_i\neq\varepsilon_{i+1}$ and $s_i=s_{i+1}$. By Proposition 5.1, the pair $(P_i(u),\gamma)$ satisfying (6.11) is unique in this case. Since $\boldsymbol{\varepsilon}$ is simple, there is at most one i such that $\varepsilon_i\neq\varepsilon_{i+1}$. Therefore, we can always make sure that $\gamma_1=\gamma_2$, completing the proof of the theorem. □

Corollary 6.9. *Conjecture 6.2 holds when $\boldsymbol{s}$ is the standard parity sequence and $\boldsymbol{\varepsilon}$ is simple.*

Proof. This follows immediately from Theorem 6.8, Theorem 2.4, and equation (4.26). □

Theorem 6.10. *Conjecture 6.2 holds for arbitrary $\boldsymbol{\varepsilon}$ when $\boldsymbol{s}$ is the standard parity sequence and $n=1$.*

Proof. The proof is similar to that of Theorem 6.8 by induction and Theorem 6.8. Again the point is to argue that γ_1 and γ_2 are related by specific rule provided the assumptions hold. □

7. Drinfeld functor and dAHA of type BC

sec:schur-weyl

7.1. **Degenerate affine Hecke algebras.** Let l be a positive integer. Following [CGM14], we first recall basics about degenerate affine Hecke algebras (dAHA for short) of types A_l and BC_l.

Denote by $\mathfrak{S}_l$ the symmetric group on l elements and set $\mathbb{Z}_2=\mathbb{Z}/2\mathbb{Z}$. Then the wreath product $\mathcal{W}_l=\mathbb{Z}_2\wr\mathfrak{S}_l$ is the Weyl group of type BC_l. Let $e_1,\cdots,e_l$ be the standard basis of $\mathbb{R}^l$, then the non-reduced root system of type BC_l consists of the following set of vectors,

$$\{\pm e_i+e_j,\ \pm e_i-e_j\mid 1\leqslant i\neq j\leqslant l\}\cup\{\pm e_i,\ \pm 2e_i\mid 1\leqslant i\leqslant l\}.$$

For $1\leqslant i\neq j\leqslant j$, let σ_{ij}, ς_i be the reflections about the root vectors e_i-e_j and e_i, respectively. Set $\sigma_i=\sigma_{i,i+1}$ for $1\leqslant i<l$.

Definition 7.1. *For $\vartheta_1\in\mathbb{C}$ and $l\in\mathbb{Z}_{>0}$, the dAHA $\mathcal{H}^l_{\vartheta_1}$ of type A_l is the associative algebra generated by the group algebra $\mathbb{C}[\mathfrak{S}_l]$ and $y_1,\cdots,y_l$ with the relations $y_iy_j=y_jy_i$, $1\leqslant i,j\leqslant l$, and*

$$\begin{aligned}&\sigma_iy_i-y_{i+1}\sigma_i=\vartheta_1, && 1\leqslant i<l,\\ &\sigma_iy_j=y_j\sigma_i, && j\neq i,i+1.\end{aligned}$$

def:dAHA-B

Definition 7.2. *For $\vartheta_1,\vartheta_2\in\mathbb{C}$ and $l\in\mathbb{Z}_{>0}$, the dAHA $\mathcal{H}^l_{\vartheta_1,\vartheta_2}$ of type BC_l is the associative algebra generated by the group algebra $\mathbb{C}[\mathcal{W}_l]$ and $y_1,\cdots,y_l$ with the relations $y_iy_j=y_jy_i$, $1\leqslant i,j\leqslant l$, and*

$$\begin{aligned}&\sigma_iy_i-y_{i+1}\sigma_i=\vartheta_1, \qquad \varsigma_ly_i=y_i\varsigma_l, \quad 1\leqslant i<l,\\ &\varsigma_ly_l+y_l\varsigma_l=\vartheta_2, \qquad \sigma_iy_j=y_j\sigma_i, \quad j\neq i,i+1.\end{aligned}$$

The following lemmas are well known, see e.g. [CGM14, Section 2].

Lemma 7.3. *The subalgebra of $\mathcal{H}^l_{\vartheta_1,\vartheta_2}$ generated by y_i, $1\leqslant i\leqslant l$, and $\mathbb{C}[\mathfrak{S}_l]$ is isomorphic to the dAHA $\mathcal{H}^l_{\vartheta_1}$ of type A_l.* □

One has the following natural embeddings,

$$\begin{aligned}&\imath_1:\mathcal{H}^{l_1}_{\vartheta_1}\hookrightarrow\mathcal{H}^l_{\vartheta_1,\vartheta_2},\quad y_i\mapsto y_i,\quad \sigma_j\mapsto\sigma_j, && 1\leqslant l_1\leqslant l,\\ &\imath_2:\mathcal{H}^{l_2}_{\vartheta_1,\vartheta_2}\hookrightarrow\mathcal{H}^l_{\vartheta_1,\vartheta_2},\quad y_i\mapsto y_{i+l-l_2},\quad \varsigma_i\mapsto\varsigma_{i+l-l_2},\quad \sigma_j\mapsto\sigma_{j+l-l_2}, && 1\leqslant l_2\leqslant l,\\ &\imath_1\otimes\imath_2:\mathcal{H}^{l_1}_{\vartheta_1}\otimes\mathcal{H}^{l_2}_{\vartheta_1,\vartheta_2}\hookrightarrow\mathcal{H}^l_{\vartheta_1,\vartheta_2}, && l_1+l_2\leqslant l.\end{aligned}\tag{7.1}$$

eq:embedding-hecke

Note that due to the relation $[\varsigma_i,y_j]=\vartheta_1\sigma_{ij}(\varsigma_i-\varsigma_j)$ for $i<j$, the last embedding $\imath_1\otimes\imath_2$ does not extend to an embedding $\mathcal{H}^{l_1}_{\vartheta_1,\vartheta_2}\otimes\mathcal{H}^{l_2}_{\vartheta_1,\vartheta_2}\hookrightarrow\mathcal{H}^l_{\vartheta_1,\vartheta_2}$.

lem:dAHA-B-other

Lemma 7.4 ([EFM09, Lemma 3.1])**.** *The algebra $\mathcal{H}^l_{\vartheta_1,\vartheta_2}$ is isomorphic to the algebra generated by elements y_i, $1\leqslant i\leqslant l$, and by $\mathbb{C}[\mathcal{W}]$ with the relations,*

$$\sigma_i\mathsf{y}_i=\mathsf{y}_{i+1}\sigma_i,\qquad \sigma_i\mathsf{y}_j=\mathsf{y}_j\sigma_i,\qquad j\neq i,i+1,$$

$$\varsigma_l\mathsf{y}_l=-\mathsf{y}_l\varsigma_l,\qquad \varsigma_l\mathsf{y}_i=\mathsf{y}_i\varsigma_l,\qquad i\neq l,$$

$$\begin{aligned}[\mathsf{y}_i,\mathsf{y}_j]=\frac{\vartheta_1\vartheta_2}{2}\sigma_{ij}(\varsigma_j-\varsigma_i)+\frac{\vartheta_1^2}{4}\sum_{\substack{k=1\\k\neq i,j}}^{l}\Big((\sigma_{jk}\sigma_{ik}-\sigma_{ik}\sigma_{jk})\\ +\sigma_{ik}\sigma_{jk}(\varsigma_i\varsigma_j-\varsigma_i\varsigma_k+\varsigma_j\varsigma_k)-\sigma_{jk}\sigma_{ik}(\varsigma_i\varsigma_j+\varsigma_i\varsigma_k-\varsigma_j\varsigma_k)\Big).\end{aligned}$$

Moreover, this presentation is related to the one in Definition 7.2 by

$$\mathsf{y}_i=y_i-\frac{\vartheta_2}{2}\varsigma_i+\frac{\vartheta_1}{2}\sum_{k=1}^{i-1}\sigma_{ik}-\frac{\vartheta_1}{2}\sum_{k=i+1}^{l}\sigma_{ik}-\frac{\vartheta_1}{2}\sum_{\substack{k=1\\k\neq i}}^{l}\sigma_{ik}\varsigma_i\varsigma_k.$$ □

Lemma 7.5 ([Lus89, 3.12])**.** *The center of the dAHA $\mathcal{H}^l_{\vartheta_1,\vartheta_2}$ (resp. $\mathcal{H}^l_{\vartheta_1}$) is generated by the $\mathfrak{S}_l$-symmetric polynomials in $y_1^2,\cdots,y_l^2$ (resp. $y_1,\cdots,y_l$).* □

7.2. **Drinfeld functor for super Yangian.** The symmetric group $\mathfrak{S}_l$ acts naturally on $V^{\otimes l}$, where the operator σ_{ij} for $i<j$ acts as

eq perm
$$\mathscr{P}^{(i,j)}=\sum_{a,b=1}^{\varkappa} s_b E^{(i)}_{ab}E^{(j)}_{ba}\in \mathrm{End}(V^{\otimes l}). \tag{7.2}$$

Here we use the standard notation

$$E^{(k)}_{ij}=1^{\otimes(k-1)}\otimes E_{ij}\otimes 1^{\otimes(l-k)}\in\mathrm{End}(V^{\otimes l}),\qquad 1\leqslant k\leqslant l.$$

Set

$$\mathbb{Q}^{(k)}=\sum_{i,j=1}^{\varkappa}(-1)^{|i||j|+|i|+|j|}E^{(k)}_{ij}\otimes E_{ij}\in\mathrm{End}(V^{\otimes l})\otimes\mathrm{End}(V),\qquad 1\leqslant k\leqslant l.$$

Let $\varepsilon=\pm1$. Let M be any $\mathcal{H}^l_{\vartheta_1}$-module. Set

$$\mathscr{D}_{\boldsymbol{s}}(M)=M\otimes V^{\otimes l},\quad \mathscr{D}^{\varepsilon}_{\boldsymbol{s}}(M)=\mathscr{D}_{\boldsymbol{s}}(M)/\sum_{i=1}^{l-1}(\mathrm{Im}\,\sigma_i-\varepsilon),$$

where the symmetric group acts on $\mathscr{D}_{\boldsymbol{s}}(M)$ by the diagonal action, namely σ_i acts on $M\otimes V^{\otimes l}$ as $\sigma_i\otimes\mathscr{P}^{(i,i+1)}$ for $1\leqslant i<l$.

For $\chi,c\in\mathbb{C}$, define

$$\mathscr{T}^{\chi}(u)=\mathscr{T}^{\chi}_1(u)\cdots\mathscr{T}^{\chi}_l(u)\in\mathcal{H}^l_{\vartheta_1}[[u^{-1}]]\otimes\mathrm{End}(V^{\otimes l})\otimes\mathrm{End}(V),$$

where

$$\mathscr{T}^{\chi}_k(u)=1+\frac{1}{u-\chi y_k+c}\otimes\mathbb{Q}^{(k)},\qquad 1\leqslant k\leqslant l.$$

Then the map $T(u)\mapsto\mathscr{T}^{\chi}(u)$ induces an action of $\mathscr{Y}_{\boldsymbol{s}}$ on $\mathscr{D}_{\boldsymbol{s}}(M)$.

The following statement for the Yangian $\mathrm{Y}(\mathfrak{gl}_N)$ case is well known, see [Ara99, Proposition 2] and [Dri86, Theorem 1].

lem:D-functor-A

Lemma 7.6 ([LM21, Lemma 4.2])**.** *Suppose $\vartheta_1\neq0$ and $\vartheta_1\chi=\varepsilon$. Let M be any $\mathcal{H}^l_{\vartheta_1}$-module. Then the map $T(u)\mapsto\mathscr{T}^{\chi}(u)$ induces an action of $\mathscr{Y}_{\boldsymbol{s}}$ on $\mathscr{D}^{\varepsilon}_{\boldsymbol{s}}(M)$.*

Therefore, one has a functor $\mathscr{D}^{\varepsilon}_{\boldsymbol{s}}$ from the category of $\mathcal{H}^l_{\vartheta_1}$-modules to the category of $\mathscr{Y}_{\boldsymbol{s}}$-modules. We call the functor $\mathscr{D}^{\varepsilon}_{\boldsymbol{s}}$ the *Drinfeld functor*. For Schur-Weyl type dualities for superalgebras of type A, see [Ser84, BR87, Moo03, Mit06, Fli20, LM21, Lu21, KL22, Lu23, She22, Jan23] for more details.

7.3. **Drinfeld functor for twisted super Yangian.** We need the following

lem:embedding

Lemma 7.7. *For any $\gamma\in\mathbb{C}$, the mapping*

eq:emd-b-new
$$\varphi: B(u)\to T(u)(G^{\varepsilon}+\gamma u^{-1})T^{-1}(-u) \tag{7.3}$$

defines a superalgebra homomorphism from the twisted Yangian $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$ to the super Yangian $\mathrm{Y}_{\boldsymbol{s}}$.

Proof. Similar to Proposition 3.3, it suffices to show that the matrix $G_{\varepsilon}+\gamma u^{-1}$ satisfies the reflection equation (3.2) which it is known in [AAC+04, RS07, BR09]. □

For brevity, we set

$$G^{\boldsymbol{\varepsilon},\gamma}(u)=G^{\boldsymbol{\varepsilon}}+\gamma u^{-1},\quad \jmath=\frac{m-n}{2}.$$

Consider the following elements in $\mathcal{H}^l_{\vartheta_1,\vartheta_2}[[u^{-1}]]\otimes\mathrm{End}(V^{\otimes l})\otimes\mathrm{End}(V)$,

$$\mathscr{T}^{\chi}_k(u)=1+\frac{1}{u-\jmath-\chi y_k}\otimes\mathbb{Q}^{(k)},\quad \mathscr{S}^{\chi}_k(u)=1-\frac{1}{u+\jmath-\chi y_k}\otimes\mathbb{Q}^{(k)},$$

for $1 \leqslant k \leqslant l$. By the identity $\mathbb{Q}^{(k)} \cdot \mathbb{Q}^{(k)} = 2\jmath \mathbb{Q}^{(k)}$, we have

eq:t-times=s
$$\mathscr{T}_k^{\chi}(u)\mathscr{S}_k^{\chi}(u) = 1. \tag{7.4}$$

Set

eq:B-chi
$$\mathscr{B}^{\chi}(u) = \mathscr{T}_1^{\chi}(u)\cdots\mathscr{T}_l^{\chi}(u)G^{\boldsymbol{\varepsilon},\gamma}(u)\mathscr{S}_l^{\chi}(-u)\cdots\mathscr{S}_1^{\chi}(-u) \tag{7.5}$$

as an element in $\mathcal{H}^l_{\vartheta_1,\vartheta_2}[[u^{-1}]] \otimes \mathrm{End}(V^{\otimes l}) \otimes \mathrm{End}(V)$. Here $G^{\boldsymbol{\varepsilon},\gamma}(u)$ stands for $1 \otimes 1 \otimes G^{\boldsymbol{\varepsilon},\gamma}(u)$.

Given any $\mathcal{H}^l_{\vartheta_1,\vartheta_2}$-module M, we can regard it as an $\mathcal{H}^l_{\vartheta_1}$-module and we have the $\mathcal{Y}_{\boldsymbol{s}}$-module $\mathscr{D}^{\varepsilon}_{\boldsymbol{s}}(M)$ if $\vartheta_1\chi = \varepsilon$, by Lemma 7.6. Moreover, the action of $\mathcal{Y}_{\boldsymbol{s}}$ on $\mathscr{D}^{\varepsilon}_{\boldsymbol{s}}(M)$ is given by

$$T(u) \mapsto \mathscr{T}_1^{\chi}(u)\cdots\mathscr{T}_l^{\chi}(u).$$

Hence it follows from Lemma 7.7 and (7.4) that $B(u) \mapsto \mathscr{B}^{\chi}(u)$ induces an action of $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$ on $\mathscr{D}^{\varepsilon}_{\boldsymbol{s}}(M)$.

The $\mathfrak{S}_l$-action on $V^{\otimes l}$ can be extended to $\mathcal{W}_l$ by setting the action of ς_k on $V^{\otimes l}$ by multiplication on the k-th factor by the matrix $G^{\boldsymbol{\varepsilon}}$. We also write this operator as G^{ε}_k. For an $\mathcal{H}^l_{\vartheta_1,\vartheta_2}$-module M, the group $\mathcal{W}_l$ acts on $M \otimes V^{\otimes l}$ by the diagonal action. We further set

$$\mathscr{D}^{\varepsilon}_{\boldsymbol{s},\boldsymbol{\varepsilon}}(M) = \mathscr{D}^{\varepsilon}_{\boldsymbol{s}}(M)/(\mathrm{Im}\,\varsigma_l - \varepsilon).$$

We shall need the following lemma. Recall that $\varpi_1 = \sum_{a=1}^{\varkappa} s_a\varepsilon_a$ and $2\jmath = \sum_{a=1}^{\varkappa} s_a$. Set

$$\mathbb{Q}^{\mathfrak{k}}_k = \sum_{i,j:\varepsilon_i=\varepsilon_j} (-1)^{|i||j|+|i|+|j|} E^{(k)}_{ij} \otimes E_{ij},$$
$$\mathbb{Q}^{\mathfrak{p}}_k = \mathbb{Q}^{(k)} - \mathbb{Q}^{\mathfrak{k}}_k = \sum_{i,j:\varepsilon_i\neq\varepsilon_j} (-1)^{|i||j|+|i|+|j|} E^{(k)}_{ij} \otimes E_{ij}.$$

lem:cgm-thm4.5

Lemma 7.8. *We have*

$$\mathbb{Q}^{\mathfrak{k}}_k\mathbb{Q}^{\mathfrak{p}}_k + \mathbb{Q}^{\mathfrak{p}}_k\mathbb{Q}^{\mathfrak{k}}_k = 2\jmath\mathbb{Q}^{\mathfrak{p}}_k,$$
$$G^{\boldsymbol{\varepsilon}}(\mathbb{Q}^{\mathfrak{k}}_k\mathbb{Q}^{\mathfrak{p}}_k - \mathbb{Q}^{\mathfrak{p}}_k\mathbb{Q}^{\mathfrak{k}}_k) = \varpi_1\mathbb{Q}^{\mathfrak{p}}_k.$$

Proof. The formulas follow from a direct computation. □

The following are the main results of this section.

prop:Drinfeld-functor-BC

Proposition 7.9. *Let M be any $\mathcal{H}^l_{\vartheta_1,\vartheta_2}$-module. If $\vartheta_2 = \vartheta_1(2\gamma + \varpi_1)$ and $\vartheta_1\chi = \varepsilon$, then the map $B(u) \mapsto \mathscr{B}^{\chi}(u)$ defines a representation of the twisted super Yangian $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$ on the space $\mathscr{D}^{\varepsilon}_{\boldsymbol{s},\boldsymbol{\varepsilon}}(M)$.*

Proof. The proof is similar to that of [CGM14, Theorem 4.5] by using Lemma 7.8. □

Therefore, one has a functor $\mathscr{D}^{\varepsilon}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$ from the category of $\mathcal{H}^l_{\vartheta_1,\vartheta_2}$-modules to the category of $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-modules. Again, we call the functor $\mathscr{D}^{\varepsilon}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$ the *Drinfeld functor*.

Let $l, l_1, l_2 \in \mathbb{Z}_{\geqslant 0}$ such that $l = l_1 + l_2$. Let M_1 be an $\mathcal{H}^{l_1}_{\vartheta_1}$-module, M_2 an $\mathcal{H}^{l_2}_{\vartheta_1,\vartheta_2}$-module. Set

$$M_1 \odot M_2 = \mathcal{H}^l_{\vartheta_1,\vartheta_2} \otimes_{\mathcal{H}^{l_1}_{\vartheta_1} \otimes \mathcal{H}^{l_2}_{\vartheta_1,\vartheta_2}} (M_1 \otimes M_2),$$

see (7.1). Then $M_1 \odot M_2$ is an $\mathcal{H}^l_{\vartheta_1,\vartheta_2}$-module and hence $\mathscr{D}^{\varepsilon}_{\boldsymbol{s},\boldsymbol{\varepsilon}}(M_1 \odot M_2)$ is $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-module.

Note that $\mathscr{D}^{\varepsilon}_{\boldsymbol{s}}(M_1)$ is a $\mathcal{Y}_{\boldsymbol{s}}$-module and $\mathscr{D}^{\varepsilon}_{\boldsymbol{s},\boldsymbol{\varepsilon}}(M_2)$ is a $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-module, thus $\mathscr{D}^{\varepsilon}_{\boldsymbol{s}}(M_1) \otimes \mathscr{D}^{\varepsilon}_{\boldsymbol{s},\boldsymbol{\varepsilon}}(M_2)$ is a $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-module induced by the coproduct in Proposition 3.5.

prop:DF-coproduct

Proposition 7.10. *As $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-modules, we have $\mathscr{D}^{\varepsilon}_{\boldsymbol{s},\boldsymbol{\varepsilon}}(M_1 \odot M_2) \cong \mathscr{D}^{\varepsilon}_{\boldsymbol{s}}(M_1) \otimes \mathscr{D}^{\varepsilon}_{\boldsymbol{s},\boldsymbol{\varepsilon}}(M_2)$.*

Proof. The proof is parallel to that of [CGM14, Porposition 4.6]. □

We say that a $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-module is *of level l* if it decomposes as direct sums of submodules over $\mathfrak{k}$ of $V^{\otimes l}$ as a $\mathfrak{k}$-module.

thm:Drinfeld-B

Theorem 7.11. *Let $\vartheta_1,\vartheta_2,\chi,\gamma$ be as in Proposition 7.9 and $p=\#\{i\,|\,\varepsilon_i=1, 1\leqslant i\leqslant \varkappa\}$. If $\max\{p,m+n-p\}<l$, then the Drinfeld functor $\mathscr{D}^{\varepsilon}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$ provides an equivalence between the category of finite-dimensional $\mathcal{H}^l_{\vartheta_1,\vartheta_2}$-modules and the category of finite-dimensional $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-modules of level l.*

We prove the theorem in Section 7.4.

thm:Drinfeld-simple

Theorem 7.12. *Let $\vartheta_1,\vartheta_2,\chi,\gamma$ be as in Proposition 7.9. Let M be an irreducible $\mathcal{H}^l_{\vartheta_1,\vartheta_2}$-module. Then $\mathscr{D}^{\varepsilon}_{\boldsymbol{s},\boldsymbol{\varepsilon}}(M)$ is either 0 or an irreducible $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$-module.*

The theorem is analogous to [Ara99, Theorem 11] for Yangian $\mathrm{Y}(\mathfrak{gl}_N)$, [Naz99, Theorem 5.5] for super Yangian of type Q_N, [LM21, Proposition 4.8] for super Yangian $\mathscr{Y}_{\boldsymbol{s}}$ and [CGM14, Theorem 4.7] for twisted Yangian of type AIII. The proof is similar to that of [CGM14, Theorem 4.7] with suitable modifications for super case as presented in the proofs of [Naz99, Theorem 5.5] and [LM21, Proposition 4.8]. Therefore, we shall not provide the details.

app:B

7.4. **Proof of Theorem 7.11.** In this section, we give a proof of Theorem 7.11. The strategy is essentially the same as in [CGM14].

Recall that the action of $\mathcal{B}_{\boldsymbol{s},\boldsymbol{\varepsilon}}$ on $\mathscr{D}^{\varepsilon}_{\boldsymbol{s},\boldsymbol{\varepsilon}}(M)$ is induced by the map $B(u)\mapsto \mathscr{B}^{\chi}(u)$, see (7.5). Expanding $\mathscr{B}^{\chi}(u)$ as a series in u^{-1} with coefficients in $\mathcal{H}^l_{\vartheta_1,\vartheta_2}\otimes \mathrm{End}(V^{\otimes l})\otimes \mathrm{End}(V)$, we find the first 3 coefficients are given by $G^{\boldsymbol{\varepsilon}}$ (understood as $1\otimes 1\otimes G^{\boldsymbol{\varepsilon}}$),

$$\gamma + 1\otimes \sum_{k=1}^{l}\big(\mathbb{Q}^{(k)}G^{\boldsymbol{\varepsilon}} + G^{\boldsymbol{\varepsilon}}\mathbb{Q}^{(k)}\big),$$

eq:B-matrix-2nd

$$\begin{aligned}&2\gamma\Big(1\otimes\sum_{k=1}^{l}\mathbb{Q}^{(k)}\Big)+\sum_{1\leqslant k<r\leqslant l}(1\otimes\mathbb{Q}^{(k)}\mathbb{Q}^{(r)})G^{\boldsymbol{\varepsilon}}+\sum_{1\leqslant r<k\leqslant l}G^{\boldsymbol{\varepsilon}}(1\otimes\mathbb{Q}^{(k)}\mathbb{Q}^{(r)})\\ &+\Big(1\otimes\sum_{k=1}^{l}\mathbb{Q}^{(k)}\Big)G^{\boldsymbol{\varepsilon}}\Big(1\otimes\sum_{r=1}^{l}\mathbb{Q}^{(r)}\Big)+\sum_{k=1}^{l}\Big(G^{\boldsymbol{\varepsilon}}\big((\jmath-\chi y_k)\otimes\mathbb{Q}^{(k)}\big)+\big((\jmath+\chi y_k)\otimes\mathbb{Q}^{(k)}\big)G^{\boldsymbol{\varepsilon}}\Big).\end{aligned}\tag{7.6}$$

We set

$$\mathscr{B}^{\chi}(u)=\sum_{r\geqslant 0}\sum_{i,j=1}^{\varkappa}(-1)^{|i||j|+|i|+|j|}\mathsf{b}_{ij}^{(r)}u^{-r}\otimes E_{ij},$$

where $\mathsf{b}_{ij}^{(r)}\in\mathcal{H}^l_{\vartheta_1,\vartheta_2}\otimes\mathrm{End}(V^{\otimes l})$. From above, we conclude that

$$\mathsf{b}_{ij}^{(0)}=\varepsilon_i\delta_{ij},\qquad \mathsf{b}_{ij}^{(1)}=\gamma\delta_{ij}+s_i(\varepsilon_i+\varepsilon_j)\sum_{k=1}^{l}1\otimes E_{ij}^{(k)}.$$

Before computing $\mathsf{b}_{ij}^{(2)}$, we prepare the following lemma.

lem:drinfeld-cal

Lemma 7.13. *Suppose $\varepsilon_i\neq\varepsilon_j$. Then as operators on $V^{\otimes l}$, we have*

eq:lem-bij2-1

$$s_i\sum_{k=1}^{l}\Big(\sum_{r=1}^{k-1}\sigma_{rk}-\sum_{r=k+1}^{l}\sigma_{rk}\Big)E_{ij}^{(k)}=-\varepsilon_i\Big(\sum_{1\leqslant k<r\leqslant l}\mathbb{Q}^{(k)}\mathbb{Q}^{(r)}G^{\boldsymbol{\varepsilon}}+\sum_{1\leqslant r<k\leqslant l}G^{\boldsymbol{\varepsilon}}\mathbb{Q}^{(k)}\mathbb{Q}^{(r)}\Big)_{ij},\tag{7.7}$$

eq:lembij2-2

$$s_i\sum_{k=1}^{l}\Big(\sum_{\substack{r=1\\ r\neq k}}\sigma_{kr}\varsigma_r\varsigma_k+\varpi_1\varsigma_k\Big)E_{ij}^{(k)}=\varepsilon_i\Big(\sum_{k,r=1}^{l}\mathbb{Q}^{(k)}G^{\boldsymbol{\varepsilon}}\mathbb{Q}^{(r)}\Big)_{ij}.\tag{7.8}$$

Here by $(\cdot)_{ij}$*, we mean the* (i,j)*-th entry, namely, for* $\mathsf{G}\in\mathrm{End}(V^{\otimes l})\otimes\mathrm{End}(V)$,

$$\mathsf{G}=\sum_{i,j=1}^{\varkappa}(-1)^{|i||j|+|j|}(\mathsf{G})_{ij}\otimes E_{ij}.$$

Proof. Recall from (7.2) that $\sigma_{rk}=\mathscr{P}^{(r,k)}=\sum_{a,b=1}^{\varkappa}s_bE_{ab}^{(r)}E_{ba}^{(k)}$, therefore the left hand of (7.7) is equal to

$$\sum_{r<k}\sum_{a=1}^{\varkappa}s_is_aE_{ia}^{(r)}E_{aj}^{(k)}-\sum_{k<r}\sum_{a=1}^{\varkappa}s_is_aE_{ia}^{(r)}E_{aj}^{(k)}.$$

A straightforward computation implies

$$\begin{aligned}\sum_{1\leqslant k<r\leqslant l}\mathbb{Q}^{(k)}\mathbb{Q}^{(r)}G^{\varepsilon}+&\sum_{1\leqslant r<k\leqslant l}G^{\varepsilon}\mathbb{Q}^{(k)}\mathbb{Q}^{(r)}\\&=\sum_{i,j,a=1}^{\varkappa}\Big(\sum_{r<k}\varepsilon_i+\sum_{k<r}\varepsilon_j\Big)E_{ia}^{(k)}E_{aj}^{(s)}\otimes E_{ij}(-1)^{|i||j|+|i|+|j|+|a|}.\end{aligned}$$

After interchanging k and r and using $\varepsilon_j=-\varepsilon_i$, one obtains (7.7).

Similarly, the left hand side of (7.8) is equal to

$$\sum_{\substack{k,r=1\\r\neq k}}^{l}\sum_{a=1}^{\varkappa}s_is_a\varepsilon_a\varepsilon_iE_{ia}^{(r)}E_{aj}^{(k)}+\varpi_1\sum_{k=1}^{l}s_i\varepsilon_iE_{ij}^{(k)},$$

while we also have

$$\begin{aligned}\sum_{k,r=1}^{l}\mathbb{Q}^{(k)}G^{\varepsilon}\mathbb{Q}^{(r)}=&\sum_{\substack{k,r=1\\r\neq k}}\sum_{i,j,a=1}^{\varkappa}\varepsilon_aE_{ia}^{(k)}E_{aj}^{(r)}\otimes E_{ij}(-1)^{|i||j|+|i|+|j|+|a|}\\&+\sum_{k=1}^{l}\sum_{i,j,a=1}^{\varkappa}\varepsilon_aE_{ij}^{(k)}\otimes E_{ij}(-1)^{|i||j|+|i|+|j|+|a|}.\end{aligned}$$

Now (7.8) follows from $\varpi_1=\sum_{a=1}^{\varkappa}s_a\varepsilon_a=\sum_{a=1}^{\varkappa}(-1)^{|a|}\varepsilon_a$. □

Note that

eq:lembij2-3
$$G^{\varepsilon}\mathbb{Q}^{(k)}+\mathbb{Q}^{(k)}G^{\varepsilon}=\sum_{i,j=1}^{\varkappa}(\varepsilon_i+\varepsilon_j)\mathbb{Q}^{(k)}.\tag{7.9}$$

It follows from (7.6), (7.9), and Lemma 7.13 that if $\varepsilon_i\neq\varepsilon_j$, then

$$\begin{aligned}s_i\mathsf{b}_{ij}^{(2)}&=2\gamma\sum_{k=1}^{l}E_{ij}^{(k)}-\varepsilon_i\varepsilon\sum_{k=1}^{l}\Big(\sum_{r=1}^{k-1}\sigma_{rk}-\sum_{r=k+1}^{l}\sigma_{rk}\Big)\otimes E_{ij}^{(k)}\\&\quad+\varepsilon_i\varepsilon\sum_{k=1}^{l}\Big(\sum_{\substack{r=1\\r\neq k}}^{l}\sigma_{kr}\varsigma_r\varsigma_k+\varpi_1\varsigma_k\Big)\otimes E_{ij}^{(k)}-2\chi\varepsilon_i\sum_{k=1}^{l}y_k\otimes E_{ij}^{(k)}\\&=-2\varepsilon_i\sum_{k=1}^{l}\Big(\chi y_k+\frac{\varepsilon}{2}\sum_{r=1}^{k-1}\sigma_{rk}-\frac{\varepsilon}{2}\sum_{r=k+1}^{l}\sigma_{rk}-\frac{\varepsilon}{2}\sum_{\substack{r=1\\r\neq k}}^{l}\sigma_{kr}\varsigma_r\varsigma_k-\frac{\varepsilon}{2}\big(\varpi_1+2\gamma\big)\varsigma_k\Big)\otimes E_{ij}^{(k)}.\end{aligned}$$

Therefore, if we suppose further that $\vartheta_2=\vartheta_1(2\gamma+\varpi_1)$ and $\varepsilon=\vartheta_1\chi$, we have

$$s_i\varepsilon\vartheta_1\mathsf{b}_{ij}^{(2)}=-2\varepsilon_i\sum_{k=1}^{l}\mathsf{y}_k\otimes E_{ij}^{(k)},$$

see Lemma 7.4 and cf. [CGM14, Equation (4.1)].

The rest of the proof is similar to the one that is outlined in [CGM14, proof of Theorem 4.3]. We shall omit the details. The $\widetilde{J}(E_{ij})$ there should be replaced by $\mathsf{b}_{ij}^{(2)}$ as the J-presentations of (twisted) super Yangians are not discussed here. The fact that the tensor space $V^{\otimes l}$ decomposes as a direct sum of irreducible modules over $\mathfrak{k} \times \mathcal{W}_l$ follows from the proof of [She22, Theorem 5.8]. A precise decomposition parallel to the one discussed in [ATY95, Introduction] (namely one only needs to change $W_{\underline{\lambda}}$ to the $\mathfrak{k}$-module associated to the tuple $\underline{\lambda}$) can be deduced from a standard approach as in [CW12, Theorem 3.11] or [ATY95], see also [Ker71, Chapter II]. This decomposition and the condition $\max\{p, \varkappa - p\} < l$ make sure that $\mathscr{D}^{\varepsilon}_{\boldsymbol{s},\boldsymbol{\varepsilon}}(M)$ is nonzero if M is nonzero.

References

[AAC$^+$04] D. Arnaudon, J. Avan, N. Crampé, A. Doikou, L. Frappat, and E. Ragoucy, *General boundary conditions for the* sl($\mathcal{N}$) *and* sl($\mathcal{M}|\mathcal{N}$) *open spin chains*, J. Stat. Mech. Theory Exp. **8** (2004), 005, 36. MR2115253 ↑35

[Ara99] T. Arakawa, *Drinfeld functor and finite-dimensional representations of Yangian*, Comm. Math. Phys. **205** (1999), no. 1, 1–18. MR1706920 ↑35, 37

[ATY95] S. Ariki, T. Terasoma, and H. Yamada, *Schur-Weyl reciprocity for the Hecke algebra of* $(\mathbf{Z}/r\mathbf{Z}) \wr S_n$, J. Algebra **178** (1995), no. 2, 374–390. MR1359891 ↑39

[BK23] L. Bagnoli and S. Kožić, *Double Yangian and reflection algebras of the Lie superalgebra* $\mathfrak{gl}_{m|n}$, arXiv preprint arXiv:2311.02410 (2023). ↑2

[BR09] S. Belliard and E. Ragoucy, *The nested Bethe ansatz for 'all' open spin chains with diagonal boundary conditions*, J. Phys. A **42** (2009), no. 20, 205203, 35. MR2515596 ↑2, 6, 7, 9, 10, 31, 35

[BR87] A. Berele and A. Regev, *Hook Young diagrams with applications to combinatorics and to representations of Lie superalgebras*, Adv. in Math. **64** (1987), no. 2, 118–175. MR884183 ↑35

[CGM14] H. Chen, N. Guay, and X. Ma, *Twisted Yangians, twisted quantum loop algebras and affine Hecke algebras of type BC*, Trans. Amer. Math. Soc. **366** (2014), no. 5, 2517–2574. MR3165646 ↑1, 2, 34, 36, 37, 39

[Che84] I. Cherednik, *Factorizing particles on a half line, and root systems*, Teoret. Mat. Fiz. **61** (1984), no. 1, 35–44. MR774205 ↑1

[CW12] S.-J. Cheng and W. Wang, *Dualities and representations of Lie superalgebras*, Graduate Studies in Mathematics, vol. 144, American Mathematical Society, Providence, RI, 2012. MR3012224 ↑2, 39

[Dri86] V. Drinfeld, *Degenerate affine Hecke algebras and Yangians*, Funktsional. Anal. i Prilozhen. **20** (1986), no. 1, 69–70. MR831053 ↑35

[EFM09] P. Etingof, R. Freund, and X. Ma, *A Lie-theoretic construction of some representations of the degenerate affine and double affine Hecke algebras of type* BC_n, Represent. Theory **13** (2009), 33–49. MR2480387 ↑34

[Fli20] Y. Flicker, *Affine quantum super Schur-Weyl duality*, Algebr. Represent. Theory **23** (2020), no. 1, 135–167. MR4058428 ↑35

[Gow07] L. Gow, *Gauss decomposition of the Yangian* $Y(\mathfrak{gl}_{m|n})$, Comm. Math. Phys. **276** (2007), no. 3, 799–825. MR2350438 ↑5, 8, 11, 13

[GR16] N. Guay and V. Regelskis, *Twisted Yangians for symmetric pairs of types B, C, D*, Math. Z. **284** (2016), no. 1-2, 131–166. MR3545488 ↑2

[Jan23] P. Jankovic, *Schur-weyl functors for super yangians and deformed double current algebras*, Ph.D thesis, University of Alberta (2023). ↑35

[Ker71] A. Kerber, *Representations of permutation groups. I*, Lecture Notes in Mathematics, vol. Vol. 240, Springer-Verlag, Berlin-New York, 1971. MR325752 ↑39

[Ket23] B. Kettle, *Orthosymplectic, Periplectic, and Twisted Super Yangians*, Ph.D thesis, University of Alberta (2023). ↑2

[KL22] J.-H. Kwon and S.-M. Lee, *Super duality for quantum affine algebras of type A*, Int. Math. Res. Not. IMRN **23** (2022), 18446–18525. MR4519149 ↑35

[LM21] K. Lu and E. Mukhin, *Jacobi-Trudi identity and Drinfeld functor for super Yangian*, Int. Math. Res. Not. IMRN **21** (2021), 16751–16810. MR4338233 ↑35, 37

[LPRS19] A. Liashyk, S. Z. Pakuliak, E. Ragoucy, and N. A. Slavnov, *New symmetries of* $\mathfrak{gl}(N)$*-invariant Bethe vectors*, J. Stat. Mech. Theory Exp. **4** (2019), 044001, 24. MR4057221 ↑8, 9

[Lu21] K. Lu, *Gelfand-Tsetlin bases of representations for super Yangian and quantum affine superalgebra*, Lett. Math. Phys. **111** (2021), no. 6, Paper No. 145, 30. MR4346502 ↑35

[Lu22] K. Lu, *A note on odd reflections of super Yangian and Bethe ansatz*, Lett. Math. Phys. **112** (2022), no. 2, Paper No. 29, 26. MR4400677 ↑2, 6, 9, 30, 32

[Lu23] K. Lu, *Schur-Weyl duality for quantum toroidal superalgebras*, J. Pure Appl. Algebra **227** (2023), no. 9, Paper No. 107382, 19. MR4569699 ↑35

[Lus89] G. Lusztig, *Affine Hecke algebras and their graded version*, J. Amer. Math. Soc. **2** (1989), no. 3, 599–635. MR991016 ↑35

[LWZ23] K. Lu, W. Wang, and W. Zhang, *A Drinfeld type Presentation of twisted Yangians*, arXiv:2308.12254 (2023). ↑2

[LWZ24] K. Lu, W. Wang, and W. Zhang, *A Drinfeld presentation of twisted Yangians via degeneration*, in preparation (2024). ↑2

[Mac02] N. MacKay, *Rational K-matrices and representations of twisted Yangians*, J. Phys. A **35** (2002), no. 37, 7865–7876. MR1945798 ↑1, 2

[Mit06] H. Mitsuhashi, *Schur-Weyl reciprocity between the quantum superalgebra and the Iwahori-Hecke algebra*, Algebr. Represent. Theory **9** (2006), no. 3, 309–322. MR2251378 ↑35

[Mol22] A. Molev, *Odd reflections in the Yangian associated with $\mathfrak{gl}(m|n)$*, Lett. Math. Phys. **112** (2022), no. 1, Paper No. 8, 15. MR4367922 ↑2, 6, 30, 32

[Mol98] A. Molev, *Finite-dimensional irreducible representations of twisted Yangians*, J. Math. Phys. **39** (1998), no. 10, 5559–5600. MR1642338 ↑23

[Moo03] D. Moon, *Highest weight vectors of irreducible representations of the quantum superalgebra $\mathfrak{U}_q(\mathrm{gl}(m,n))$*, J. Korean Math. Soc. **40** (2003), no. 1, 1–28. MR1945710 ↑35

[MR02] A. Molev and E. Ragoucy, *Representations of reflection algebras*, Rev. Math. Phys. **14** (2002), no. 3, 317–342. MR1894013 ↑1, 2, 9, 10, 11, 13, 20, 30

[Naz20] M. Nazarov, *Yangian of the general linear Lie superalgebra*, SIGMA Symmetry Integrability Geom. Methods Appl. **16** (2020), Paper No. 112, 24 pages. MR4170710 ↑5

[Naz91] M. Nazarov, *Quantum Berezinian and the classical Capelli identity*, Lett. Math. Phys. **21** (1991), no. 2, 123–131. MR1093523 ↑3

[Naz99] M. Nazarov, *Yangian of the queer Lie superalgebra*, Comm. Math. Phys. **208** (1999), no. 1, 195–223. MR1729884 ↑37

[Ols92] G. Olshanski, *Twisted Yangians and infinite-dimensional classical Lie algebras*, Quantum groups (Leningrad, 1990), 1992, pp. 104–119. MR1183482 ↑2

[Pen16] Y.-N. Peng, *Parabolic presentations of the super Yangian $Y(\mathfrak{gl}_{M|N})$ associated with arbitrary 01-sequences*, Comm. Math. Phys. **346** (2016), no. 1, 313–347. MR3528423 ↑5, 8

[RS07] E. Ragoucy and G. Satta, *Analytical Bethe ansatz for closed and open $\mathrm{gl}(\mathcal{M}|\mathcal{N})$ super-spin chains in arbitrary representations and for any Dynkin diagram*, J. High Energy Phys. **9** (2007), 001, 40. MR2342430 ↑2, 6, 7, 9, 10, 35

[Ser84] A. Sergeev, *Tensor algebra of the identity representation as a module over the Lie superalgebras $\mathrm{Gl}(n,\, m)$ and $Q(n)$*, Mat. Sb. (N.S.) **123(165)** (1984), no. 3, 422–430. MR735715 ↑35

[She22] Y. Shen, *Quantum supersymmetric pairs and ıSchur duality of type AIII*, arXiv:2210.01233 (2022). ↑12, 35, 39

[Skl88] E. Sklyanin, *Boundary conditions for integrable quantum systems*, J. Phys. A **21** (1988), no. 10, 2375–2389. MR953215 ↑1

[Zha95] R. B. Zhang, *Representations of super Yangian*, J. Math. Phys. **36** (1995), no. 7, 3854–3865. MR1339907 ↑2, 32

[Zha96] R. B. Zhang, *The $\mathrm{gl}(M|N)$ super Yangian and its finite-dimensional representations*, Lett. Math. Phys. **37** (1996), no. 4, 419–434. MR1401045 ↑2, 5

K.L.: Department of Mathematics, University of Virginia,
141 Cabell Dr, Charlottesville, VA 22903, USA
Email address: kang.lu@virginia.edu