ON BETHE EIGENVECTORS AND HIGHER TRANSFER MATRICES FOR SUPERSYMMETRIC SPIN CHAINS

KANG LU

ABSTRACT. We study the $\mathfrak{gl}_{m|n}$ XXX spin chains defined on tensor products of highest $\mathfrak{gl}_{m|n}$-modules. We show that the on-shell Bethe vectors are eigenvectors of higher transfer matrices and compute the corresponding eigenvalues, confirming [LM21b, Conjecture 5.15] and extending the main result of [MTV06] to supersymmetric case. We then take the classical limits and obtain the corresponding results for the $\mathfrak{gl}_{m|n}$ Gaudin models.

Keywords: XXX spin chains, Gaudin models, Bethe ansatz, super Yangians, Berezinian

1. INTRODUCTION

In this paper, we study the quantum integrable models associated with the Lie superalgebra $\mathfrak{gl}_{m|n}$, the XXX spin chains and Gaudin models. An important problem in the study of quantum integrable systems is to find the eigenvectors and the corresponding eigenvalues of the Hamiltonians.

Let $m, n \in \mathbb{Z}_{\geqslant 0}$ and set $N = m + n$. Let $T_{ab}(u)$, $1 \leqslant a, b \leqslant N$, be the RTT generating series of the super Yangian $\mathrm{Y}(\mathfrak{gl}_{m|n})$ associated with $\mathfrak{gl}_{m|n}$ and $T(u) = \sum_{a,b=1}^{N} E_{ab} \otimes T_{ab}(u)$ the monodromy matrix. The integral of motions for XXX spin chains are given by the quantum Berezinian (superdeterminant) of a certain Manin matrix, see [MR14],

$$\mathrm{Ber}(1 - T^{\dagger}(u) Q e^{-\partial_u}) = 1 + \sum_{k=1}^{\infty} (-1)^k \mathcal{T}_{k,Q}(u) e^{-k\partial_u},$$

where $Q = \mathrm{diag}(Q_1, \dots, Q_N)$ is invertible and $\dagger : E_{ab} \mapsto (-1)^{|a||b|+|a|} E_{ba}$ is the super transpose. The series $\mathcal{T}_{k,Q}(u)$, whose coefficients are elements in the super Yangian $\mathrm{Y}(\mathfrak{gl}_{m|n})$, are called *transfer matrices*. The subalgebra of $\mathrm{Y}(\mathfrak{gl}_{m|n})$ generated by the coefficients of $\mathcal{T}_{k,Q}(u)$ for all $k > 0$ is commutative and called the Bethe subalgebra, cf. [NO96].

Given a highest weight representation of $\mathrm{Y}(\mathfrak{gl}_{m|n})$, we are interested in finding the joint eigenvectors of transfer matrices. The same problem for diagonalizing the standard transfer matrix $\mathcal{T}_{1,Q}(u)$ has been studied in [BR08] using algebraic Bethe ansatz [STF79, TF79] and nested Bethe ansatz [KR83]. The main result of [BR08] indicates that if the sequence of parameters $\boldsymbol{t}$ satisfies the Bethe ansatz equation, then one can construct a Bethe vector $\mathbb{B}(\boldsymbol{t})$ which (if nonzero) is an eigenvector of $\mathcal{T}_{1,Q}(u)$. Moreover, the corresponding eigenvalue can be computed explicitly. Following [MTV06], we extend the results of [BR08] to show that the Bethe vector $\mathbb{B}(\boldsymbol{t})$ is indeed a joint eigenvector of all transfer matrices $\mathcal{T}_{k,Q}(u)$, $k \in \mathbb{Z}_{>0}$, and give the explicit eigenvalues for each transfer matrix, cf. [Tsu97].

Let me explain our results in more detail. Let M be a highest weight representation of $\mathrm{Y}(\mathfrak{gl}_{m|n})$ generated by a nonzero vector v such that $T_{aa}(u)v = \zeta_a(u)v$ and $T_{ab}(u)v = 0$ for all $1 \leqslant b < a \leqslant N$, where $\zeta_a(u)$ is an arbitrary power series of u^{-1} in $1 + u^{-1}\mathbb{C}[[u^{-1}]]$. Let $\boldsymbol{\xi} = (\xi^1, \dots, \xi^{N-1})$ be a sequence of nonnegative integers and $\boldsymbol{t} = (t^1_1, \dots, t^1_{\xi^1}; \dots; t^{N-1}_1, \dots, t^{N-1}_{\xi^{N-1}})$ a sequence of complex numbers. Set $y_a(u)=\prod_{i=1}^{\xi^a}(u-t_i^a)$, $1\leqslant a<N$, and $y_0(u)=y_N(u)=1$. Let $\kappa_a=1$ if $1\leqslant a\leqslant m$ and $\kappa_a=-1$ if $m<a\leqslant N$. The Bethe ansatz equation for XXX spin chain associated with $\mathfrak{gl}_{m|n}$ is a system of algebraic equations in $\boldsymbol{t}$,

$$-\frac{\kappa_a Q_a}{\kappa_{a+1}Q_{a+1}}\frac{\zeta_a(t_i^a)}{\zeta_{a+1}(t_i^a)}\frac{y_{a-1}(t_i^a+\kappa_a)}{y_{a-1}(t_i^a)}\frac{y_a(t_i^a-\kappa_a)}{y_a(t_i^a+\kappa_{a+1})}\frac{y_{a+1}(t_i^a)}{y_{a+1}(t_i^a-\kappa_{a+1})}=1,$$

for $1\leqslant a<N$ and $1\leqslant i\leqslant \xi^a$, see (3.18). Suppose $\boldsymbol{t}$ satisfies the Bethe ansatz equation and let $\mathbb{B}^v_{\boldsymbol{\xi}}(\boldsymbol{t})\in M$ be the corresponding (on-shell) Bethe vector, see e.g. (3.8), (3.17). Then the eigenvalues of the transfer matrices acting on $\mathbb{B}^v_{\boldsymbol{\xi}}(\boldsymbol{t})$ can be compactly described as follows,

$$\mathrm{Ber}(1-QT(u)e^{-\partial_u})\mathbb{B}^v_{\boldsymbol{\xi}}(\boldsymbol{t})=\mathbb{B}^v_{\boldsymbol{\xi}}(\boldsymbol{t})\prod_{1\leqslant a\leqslant N}^{\longrightarrow}\Big(1-Q_a\zeta_a(u)\frac{y_{a-1}(u+\kappa_a)y_a(u-\kappa_a)}{y_{a-1}(u)y_a(u)}e^{-\partial_u}\Big)^{\kappa_a},$$

see Corollaries 3.4, 3.6, which follows from the main technical results, Theorem 3.3. Such a statement was previously established for the case $n=0$ in [MTV06], and for the case $m=n=1$ in [LM21a, Theorem 6.4] by a brute force computation.

We show it for the case when M is a tensor product of evaluation highest weight modules. Such a rational difference operator also appeared in [Tsu97] when M is a tensor product of evaluation vector representations. However, the Bethe vector is not discussed there. Note that the same approach also works for the full generality.

We prove Theorem 3.3 by induction on m. For the base case of $\mathfrak{gl}_{0|n}$, it is essentially [MTV06, Theorem 5.2] by the correspondences between transfer matrices and Bethe vectors for $\mathrm{Y}(\mathfrak{gl}_{m|n})$ and $\mathrm{Y}(\mathfrak{gl}_{n|m})$, see Section 4.2. Then we perform the nested Bethe ansatz and use induction hypothesis. Since our induction is on m, the first index is always even. Therefore the procedure for nested Bethe ansatz turns out to be similar to the case of the nonsuper case as in [MTV06].

By taking the classical limits, we obtain the corresponding statement for the Gaudin models associated with $\mathfrak{gl}_{m|n}$. Explicitly, let $M_1,\dots,M_\ell$ be highest weight $\mathfrak{gl}_{m|n}$-modules with highest weights $\Lambda_1,\dots,\Lambda_\ell$, where $\Lambda_i=(\Lambda_i^1,\dots,\Lambda_i^N)$. Let $\boldsymbol{z}=(z_1,\dots,z_\ell)$ be a sequence of pairwise distinct complex numbers. Let $\boldsymbol{\xi}$, $\boldsymbol{t}$, and $y_a(u)$, $0\leqslant a\leqslant N$, be as before. The Bethe ansatz equation for Gaudin models associated with $\mathfrak{gl}_{m|n}$ is a system of algebraic equations in $\boldsymbol{t}$,

$$K_a-K_{a+1}+\sum_{j=1}^{\ell}\frac{\kappa_a\Lambda_j^a-\kappa_{a+1}\Lambda_j^{a+1}}{t_i^a-z_j}+\frac{\kappa_a y'_{a-1}(t_i^a)}{y_{a-1}(t_i^a)}-\frac{(\kappa_a+\kappa_{a+1})y''_a(t_i^a)}{2y'_a(t_i^a)}+\frac{\kappa_{a+1}y'_{a+1}(t_i^a)}{y_{a+1}(t_i^a)}=0,$$

for $1\leqslant a<N$ and $1\leqslant i\leqslant\xi^a$, see (5.4). Suppose $\boldsymbol{t}$ satisfies the Bethe ansatz equation and let $\mathbb{F}^v_{\boldsymbol{\xi}}(\boldsymbol{t})\in M_1\otimes\cdots\otimes M_\ell$ be the corresponding (on-shell) Bethe vector. The Gaudin transfer matrices (higher Gaudin Hamiltonians) are elements of the universal enveloping superalgebra $\mathrm{U}(\mathfrak{gl}_{m|n}[x])$ of the current superalgebra $\mathfrak{gl}_{m|n}[x]$ which are again given by the quantum Berezinian, see [MR14, HM20],

$$\mathrm{Ber}(\partial_u-K-L^{\dagger}(u))=\sum_{r=0}^{\infty}\mathcal{G}_{r,K}(u)\partial_u^{m-n-r},$$

where $L(u)$ is the generating matrix of $\mathfrak{gl}_{m|n}[x]$ and $K = \mathrm{diag}(K_1, \dots, K_N)$. Then the eigenvalues of the Gaudin transfer matrices acting on $\mathbb{F}^v_{\boldsymbol{\xi}}(\boldsymbol{t})$ can be compactly described as follows,

$$\mathrm{Ber}(\partial_u - K - L^\dagger(u))\,\mathbb{F}^v_{\boldsymbol{\xi}}(\boldsymbol{t}) = \mathbb{F}^v_{\boldsymbol{\xi}}(\boldsymbol{t}) \prod_{1\leqslant a\leqslant N}^{\longrightarrow} \Big(\partial_u - K_a - \kappa_a\Big(\sum_{j=1}^{\ell} \frac{\Lambda_j^a}{u-z_j} + \frac{y'_{a-1}(u)}{y_{a-1}(u)} - \frac{y'_a(u)}{y_a(u)}\Big)\Big)^{\kappa_a},$$

see Theorem 5.2.

Finally, we remark that the results can be generalized beyond the standard parity sequence (root system). Indeed, for a super Yangian of type A whose first index is odd, in (4.5) and (4.6), one can use symmetric power instead of anti-symmetric power. Alternatively, one can introduce the shift parameter for the super Yangian. Changing parity of the fundamental space is related to negating the shift parameter. An interesting question is as follows: given a finite-dimensional irreducible representation M of $\mathrm{Y}(\mathfrak{gl}_{m|n})$, how a Bethe vector associated with a solution $\boldsymbol{t}$ corresponds to the Bethe vector associated with another solution $\tilde{\boldsymbol{t}}$ obtained from $\boldsymbol{t}$ by the fermionic reproduction procedure (odd reflection), see e.g. [HMVY19, HLM19, Mol22, Lu22a]. When the twisting matrix Q is regular simple, one would expect that they are proportional. When the representation is the tensor product of evaluation polynomial module and Q is the identity matrix, they are related by a simple Lie superalgebra action, see [HMVY19, Corollary 5.6]. For more general cases, the relation remains unclear, see a related discussion in [HMVY19, Section 8.3]. It is also well known that in general Bethe eigenvectors obtained from this approach does not provide the full list of eigenvectors of the Bethe subalgebra. A complete list of eigenvectors should be described using the Q-system, see [MV17, HLM19, GJS22]. Results on the completeness of Bethe ansatz for supersymmetric XXX spin chains can be found in [LM21a, CLV22].

The paper is organized as follows. We recall the basics of the Lie superalgebra $\mathfrak{gl}_{m|n}$, the corresponding Yangian, and the fusion procedure in Section 2. Section 3 is devoted to the study of XXX spin chains where the main results for XXX spin chains are given. We prove the main technical result Theorem 3.3 in Section 4. We discuss Gaudin models and the corresponding main results in Section 5. By taking the classical limits, we prove the main results for Gaudin models in Section 6.

Acknowledgement. The author thanks E. Mukhin and V. Tarasov for valuable discussions.

2. Preliminaries

2.1. **Basics.** Throughout this article we will use the following general conventions.

A *vector superspace* $W = W_{\bar 0} \oplus W_{\bar 1}$ is a $\mathbb{Z}_2$-graded vector space. We call elements of $W_{\bar 0}$ *even* and elements of $W_{\bar 1}$ *odd*. We write $|w| \in \{\bar 0, \bar 1\}$ for the parity of a homogeneous element $w \in W$. Whenever $|v|$ is used, we implicitly assume that v is homogeneous. Set $(-1)^{\bar 0} = 1$ and $(-1)^{\bar 1} = -1$.

Let A and B be associative superalgebras (namely $\mathbb{Z}_2$-graded algebras). Then their tensor product $A \otimes B$ is also an associative superalgebra with the structure given by

$$(a \otimes b)(a' \otimes b') = aa' \otimes bb'(-1)^{|a'||b|}, \quad |a \otimes b| = |a| + |b|,$$

for any elements $a, a' \in A$ and $b, b' \in B$.

For any $\mathbb{Z}_2$-graded modules V and W over A and B, respectively, the vector superspace $V \otimes W$ is a $\mathbb{Z}_2$-graded module over $A \otimes B$ with the structure given by

$$(a \otimes b)(v \otimes w) = av \otimes bw(-1)^{|b||v|}, \quad |v \otimes w| = |v| + |w|$$

for any elements $v \in V$ and $w \in W$.

A superalgebra *homomorphism* $\alpha : A \to B$ is a linear map satisfying $\alpha(aa') = \alpha(a)\alpha(a')$ for all $a, a' \in A$. A superalgebra *antihomomorphism* $\beta : A \to B$ is a linear map satisfying $\beta(aa') = \beta(a')\beta(a)(-1)^{|a||a'|}$ for all $a, a' \in A$.

We use the standard superscript notation for embeddings of tensor factors into tensor products. If $A_1, \ldots, A_k$ are unital associative superalgebras, and $a \in A_i$, then

$$a^{(i)} = 1^{\otimes(i-1)} \otimes a \otimes 1^{\otimes(k-i)} \in A_1 \otimes \cdots \otimes A_k.$$

If $a \in A_i$ and $b \in A_j$, then $(a \otimes b)^{(ij)} = a^{(i)} b^{(j)}$, etc. For example, if $k = 2$ and $A_1 = A_2 = A$, then

$$a^{(1)} = a \otimes 1, \quad b^{(2)} = 1 \otimes b, \quad (a \otimes b)^{(12)} = a \otimes b, \quad (a \otimes b)^{(21)} = b \otimes a(-1)^{|a||b|}.$$

For products of noncommuting factors, we use the following notation:

$$\overrightarrow{\prod_{1\leqslant i\leqslant k}} X_i = X_1 \cdots X_k, \qquad \overleftarrow{\prod_{1\leqslant i\leqslant k}} X_i = X_k \cdots X_1,$$

$$\overrightarrow{\prod_{1\leqslant i<j\leqslant k}} = \overrightarrow{\prod_{1\leqslant i\leqslant k}} \; \overrightarrow{\prod_{i<j\leqslant k}}, \qquad \overleftarrow{\prod_{1\leqslant i<j\leqslant k}} = \overleftarrow{\prod_{1\leqslant j\leqslant k}} \; \overleftarrow{\prod_{1\leqslant i<j}}.$$

2.2. **Lie superalgebra $\mathfrak{gl}_{m|n}$.** Fix $m, n \in \mathbb{Z}_{\geqslant 0}$. Set $N = m + n$ and $\mathcal{N} = m - n$. Let $\mathbb{C}^{m|n}$ be a complex vector superspace, with $\dim(\mathbb{C}^{m|n})_{\bar{0}} = m$ and $\dim(\mathbb{C}^{m|n})_{\bar{1}} = n$. Note that $\dim(\mathbb{C}^{m|n}) = N$ and $\mathrm{sdim}(\mathbb{C}^{m|n}) = \mathcal{N}$. Choose a homogeneous basis $\mathbf{v}_a$, $1 \leqslant a \leqslant N$, of $\mathbb{C}^{m|n}$ such that $|\mathbf{v}_a| = \bar{0}$ for $1 \leqslant a \leqslant m$ and $|\mathbf{v}_a| = \bar{1}$ for $m < a \leqslant N$. We call it the *standard basis* of $\mathbb{C}^{m|n}$. Set $|a| = |\mathbf{v}_a|$.

Denote $\mathbb{C}^{m|n}$ by $\mathcal{V}$ and consider elements of $\mathrm{End}(\mathcal{V})$ as (super)matrices with respect to the standard basis of $\mathcal{V}$. In particular, we have the matrix units E_{ab} such that $E_{ab}\mathbf{v}_c = \delta_{bc}\mathbf{v}_a$ for $1 \leqslant a, b, c \leqslant N$.

The Lie superalgebra $\mathfrak{gl}_{m|n}$ is generated by elements e_{ab}, $1 \leqslant a, b \leqslant N$, with the supercommutator relations

$$[e_{ab}, e_{cd}] = \delta_{bc} e_{ad} - (-1)^{(|a|+|b|)(|c|+|d|)} \delta_{ad} e_{cb}, \tag{2.1}$$

and the parity of e_{ab} is given by $|a| + |b|$. We have the standard nilpotent subalgebras of $\mathfrak{gl}_{m|n}$,

$$\mathfrak{n}_+ = \bigoplus_{1\leqslant a<b\leqslant N} \mathbb{C} e_{ab}, \quad \mathfrak{n}_- = \bigoplus_{1\leqslant a<b\leqslant N} \mathbb{C} e_{ba}.$$

A vector v in a $\mathfrak{gl}_{m|n}$-module is a *weight vector of weight* $(\Lambda^1, \ldots, \Lambda^N)$ if $e_{aa}v = \Lambda^a v$ for $1 \leqslant a \leqslant N$. A vector v is called a *singular vector* if $e_{ab}v = 0$ for any $1 \leqslant a < b \leqslant N$.

Let $P = \sum_{a,b=1}^{N} E_{ab} \otimes E_{ba}(-1)^{|b|}$ which is the super flip operator: $P(v \otimes w) = (-1)^{|v||w|} w \otimes v$. Clearly, we have $X^{(21)} = PXP$ for $X \in \mathrm{End}(\mathcal{V}^{\otimes 2})$.

Let S_k be the symmetric group permuting $\{1, 2, \ldots, k\}$ with the simple permutation $\sigma_i = (i, i+1)$ for $1 \leqslant i < k$. The symmetric group acts on $\mathcal{V}^{\otimes k}$ by $\sigma_i \mapsto P^{(i,i+1)}$. Let $\mathbb{H}_{\{k\}}$, $\mathbb{A}_{\{k\}} \in \mathrm{End}(\mathcal{V}^{\otimes k})$ be the symmetrizer and anti-symmetrizer, respectively,

$$\mathbb{H}_{\{k\}} = \frac{1}{k!}\sum_{\sigma\in S_k}\sigma, \quad \mathbb{A}_{\{k\}} = \frac{1}{k!}\sum_{\sigma\in S_k}\mathrm{sign}(\sigma)\sigma,$$

where σ is identified as the corresponding operator in $\mathrm{End}(\mathcal{V}^{\otimes k})$. Denote by $\mathcal{V}^{\wedge k}$ the image of $\mathbb{A}_{\{k\}}$. For any even matrix $Q \in \mathrm{End}(\mathcal{V})$, set $Q^{\wedge k} = Q^{\otimes k}\big|_{\mathcal{V}^{\wedge k}}$.

The map $\mathfrak{gl}_{m|n} \to \mathrm{End}(\mathcal{V}),\ e_{ab} \mapsto E_{ab}$ defines a $\mathfrak{gl}_{m|n}$-module structure on $\mathcal{V}$. We call it the *vector representation* of $\mathfrak{gl}_{m|n}$. The space $\mathcal{V}^{\wedge k}$ is also a $\mathfrak{gl}_{m|n}$-module with the action $e_{ab} \mapsto (E_{ab}^{(1)} + \cdots + E_{ab}^{(k)})\big|_{\mathcal{V}^{\wedge k}}$.

Define a *supertrace* $\mathrm{str} : \mathrm{End}(\mathcal{V}) \to \mathbb{C}$, which is supercyclic,

$$\mathrm{str}(E_{ab}) = (-1)^{|b|}\delta_{ab}, \qquad \mathrm{str}([E_{ab}, E_{cd}]) = 0. \tag{2.2}$$

Define the supertranspose $\dagger$,

$$\dagger : \mathrm{End}(\mathcal{V}) \to \mathrm{End}(\mathcal{V}), \quad E_{ab} \mapsto (-1)^{|a||b|+|a|}E_{ba}.$$

The supertranspose is an anti-homomorphism and respects the supertrace,

$$(AB)^\dagger = (-1)^{|A||B|}B^\dagger A^\dagger, \qquad \mathrm{str}(A) = \mathrm{str}(A^\dagger), \tag{2.3}$$

for all supermatrices A, B.

2.3. **Super Yangian.** Define the *rational R-matrix* $R(u) = u + P \in \mathrm{End}(\mathcal{V}^{\otimes 2})$ which satisfies the Yang-Baxter equation

$$R^{(12)}(u-v)R^{(13)}(u)R^{(23)}(v) = R^{(23)}(v)R^{(13)}(u)R^{(12)}(u-v). \tag{2.4}$$

The *super Yangian* $\mathrm{Y}(\mathfrak{gl}_{m|n})$ is a unital associative superalgebra with generators $T_{ab}^{\{s\}}$ of parity $|a|+|b|$, $1 \leqslant a, b \leqslant N$, $s \in \mathbb{Z}_{>0}$. Consider the generating series

$$T_{ab}(u) = \delta_{ab} + \sum_{s=1}^{\infty} T_{ab}^{\{s\}}u^{-s}$$

and combine the series into a linear operator $T(u) = \sum_{a,b=1}^{N} E_{ab}\otimes T_{ab}(u) \in \mathrm{End}(\mathcal{V})\otimes\mathrm{Y}(\mathfrak{gl}_{m|n})[[u^{-1}]]$. The defining relations of $\mathrm{Y}(\mathfrak{gl}_{m|n})$ are given by

$$R^{(12)}(u-v)T^{(13)}(u)T^{(23)}(v) = T^{(23)}(v)T^{(13)}(u)R^{(12)}(u-v). \tag{2.5}$$

Alternatively, the defining relation (2.5) gives

$$\begin{aligned}(u-v)[T_{ab}(u), T_{cd}(v)] &= (-1)^{|a||c|+|a||d|+|c||d|}\big(T_{cb}(v)T_{ad}(u) - T_{cb}(u)T_{ad}(v)\big)\\ &= (-1)^{|a||b|+|a||d|+|b||d|}\big(T_{ad}(u)T_{cb}(v) - T_{ad}(v)T_{cb}(u)\big).\end{aligned} \tag{2.6}$$

The super Yangian $\mathrm{Y}(\mathfrak{gl}_{m|n})$ is a Hopf superalgebra with a coproduct and an opposite coproduct given by

$$\begin{aligned}\Delta : T_{ab}(u) &\mapsto \sum_{c=1}^{N} T_{cb}(u)\otimes T_{ac}(u), \qquad 1 \leqslant a, b \leqslant N,\\ \widetilde{\Delta} : T_{ab}(u) &\mapsto \sum_{c=1}^{N}(-1)^{(|a|+|c|)(|c|+|b|)}T_{ac}(u)\otimes T_{cb}(u),\end{aligned} \tag{2.7}$$

which have equivalent matrix forms

$$\begin{aligned}(\mathrm{id}\otimes\Delta)(T(u)) &= T^{(13)}(u)T^{(12)}(u),\\ (\mathrm{id}\otimes\widetilde{\Delta})(T(u)) &= T^{(12)}(u)T^{(13)}(u).\end{aligned} \tag{2.8}$$

For any complex number $z\in\mathbb{C}$, there is an automorphism

$$\rho_z : \mathrm{Y}(\mathfrak{gl}_{m|n}) \to \mathrm{Y}(\mathfrak{gl}_{m|n}), \qquad T_{ab}(u) \to T_{ab}(u-z), \tag{2.9}$$

where $(u-z)^{-1}$ is expanded as a power series in u^{-1}. The *evaluation homomorphism* is defined by the rule:

$$\epsilon : \mathrm{Y}(\mathfrak{gl}_{m|n}) \to \mathrm{U}(\mathfrak{gl}_{m|n}), \qquad T_{ba}^{\{s\}} \mapsto (-1)^{|a|}\delta_{1s}e_{ab}, \tag{2.10}$$

for $s\in\mathbb{Z}_{>0}$. The super Yangian $\mathrm{Y}(\mathfrak{gl}_{m|n})$ contains $\mathrm{U}(\mathfrak{gl}_{m|n})$ as a Hopf subalgebra via the embedding given by $e_{ab}\mapsto(-1)^{|a|}T_{ba}^{\{1\}}$. By (2.6), one has

$$[T_{ab}^{\{1\}}, T_{cd}(x)] = (-1)^{|a||c|+|a||d|+|c||d|}\big(\delta_{ad}T_{cb}(u) - \delta_{cb}T_{ad}(u)\big). \tag{2.11}$$

The relation (2.11) implies that

$$[E_{ab}\otimes 1 + 1\otimes e_{ab}, T(x)] = 0, \tag{2.12}$$

for any $1\leqslant a,b\leqslant N$.

For any $\mathfrak{gl}_{m|n}$-module M and $z\in\mathbb{C}$, denote by $M(z)$ the $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module obtained by pulling back of M through the homomorphism $\pi(z) := \epsilon\circ\rho_z$. The module $M(z)$ is called an *evaluation module at the evaluation point* z.

Let $\mathrm{Y}_+(\mathfrak{gl}_{m|n})$ be the left ideal of $\mathrm{Y}(\mathfrak{gl}_{m|n})$ generated by the coefficients of the series $T_{ba}(u)$ for $1\leqslant a<b\leqslant N$. Let θ be the canonical projection $\theta : \mathrm{Y}(\mathfrak{gl}_{m|n}) \to \mathrm{Y}(\mathfrak{gl}_{m|n})/\mathrm{Y}_+(\mathfrak{gl}_{m|n})$. For $A,B\in\mathrm{Y}(\mathfrak{gl}_{m|n})$, we write $A\simeq B$ if $\theta(A)=\theta(B)$. In other words, $A\simeq B$ if $A-B\in\mathrm{Y}_+(\mathfrak{gl}_{m|n})$. The following lemma is straightforward by (2.6) and (2.7).

Lemma 2.1. *For any $1\leqslant a,b\leqslant N$, we have*

(1) *the coefficients of the series $[T_{aa}(u),T_{bb}(v)]$ are in $\mathrm{Y}_+(\mathfrak{gl}_{m|n})$;*
(2) *if $Z\in\mathrm{Y}_+(\mathfrak{gl}_{m|n})$, then the coefficients of $ZT_{aa}(u)$ are also in $\mathrm{Y}_+(\mathfrak{gl}_{m|n})$;*
(3) *the coefficients of $\Delta(T_{aa}(u)) - T_{aa}(u)\otimes T_{aa}(u)$ and $\widetilde{\Delta}(T_{aa}(u)) - T_{aa}(u)\otimes T_{aa}(u)$ are in $\mathrm{Y}_+(\mathfrak{gl}_{m|n})\otimes\mathrm{Y}(\mathfrak{gl}_{m|n}) + \mathrm{Y}(\mathfrak{gl}_{m|n})\otimes\mathrm{Y}_+(\mathfrak{gl}_{m|n})$.*

We say that a vector v in a $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module is a *singular ℓ-weight vector* if $\mathrm{Y}_+(\mathfrak{gl}_{m|n})v=0$ and

$$T_{aa}(u)v = \lambda_a(u)v, \qquad \lambda_a(u)\in 1+u^{-1}\mathbb{C}[u^{-1}].$$

In this case, we call $\boldsymbol{\lambda}(u) = (\lambda_1(u),\dots,\lambda_N(u))$ the *ℓ-weight* of v.

2.4. **Fusion procedure.** We recall the R-matrices defined by fusion procedure and their properties which will be used to define higher transfer matrices. Since the proofs of these statements are parallel to the even case, see [MTV06], [Mol07, Propositions 1.6.2 & 1.6.3], and references therein, we shall omit the details.

Lemma 2.2. *We have*

$$\prod_{1\leqslant i<j\leqslant k}^{\longrightarrow} R^{(ij)}(j-i)=\prod_{1\leqslant i<j\leqslant k}^{\longleftarrow} R^{(ij)}(j-i)=\mathbb{H}_{\{k\}}\prod_{j=1}^{k} j^{k-j+1},$$

$$\prod_{1\leqslant i<j\leqslant k}^{\longrightarrow} R^{(ij)}(i-j)=\prod_{1\leqslant i<j\leqslant k}^{\longleftarrow} R^{(ij)}(i-j)=\mathbb{A}_{\{k\}}(-1)^k\prod_{j=1}^{k}(-j)^{k-j+1}.$$

By the Yang-Baxter equation (2.4) and Lemma 2.2, one has

$$\begin{aligned}\mathbb{A}_{\{k\}}^{(1\dots k)}\mathbb{A}_{\{l\}}^{(k+1,\dots,k+l)}&\prod_{1\leqslant i\leqslant k}^{\longrightarrow}\prod_{1\leqslant j\leqslant l}^{\longleftarrow} R^{(i,j+k)}(u+i-j-k+l)\\ &=\Big(\prod_{1\leqslant i\leqslant k}^{\longleftarrow}\prod_{1\leqslant j\leqslant l}^{\longrightarrow} R^{(i,j+k)}(u+i-j-k+l)\Big)\mathbb{A}_{\{k\}}^{(1\dots k)}\mathbb{A}_{\{l\}}^{(k+1,\dots,k+l)}.\end{aligned}\tag{2.13}$$

Define $R^{\wedge k,\wedge l}(u)$ acting on $\mathcal{V}^{\wedge k}\otimes\mathcal{V}^{\wedge l}$ by

$$R^{\wedge k,\wedge l}(u)=\prod_{1\leqslant i\leqslant k}^{\longleftarrow}\prod_{1\leqslant j\leqslant l}^{\longrightarrow} R^{(i,j+k)}(u+i-j-k+l)\big|_{\mathcal{V}^{\wedge k}\otimes\mathcal{V}^{\wedge l}}\in\mathrm{End}(\mathcal{V}^{\wedge k})\otimes\mathrm{End}(\mathcal{V}^{\wedge l}).$$

We have the following properties for these R-matrices,

$$\begin{aligned}\big(R^{\wedge k,\wedge l}(u-v)\big)^{(12)}&\big(R^{\wedge k,\wedge \ell}(u)\big)^{(13)}\big(R^{\wedge l,\wedge \ell}(v)\big)^{(23)}\\ &=\big(R^{\wedge l,\wedge \ell}(v)\big)^{(23)}\big(R^{\wedge k,\wedge \ell}(u)\big)^{(13)}\big(R^{\wedge k,\wedge l}(u-v)\big)^{(12)},\end{aligned}$$

and

$$\big[R^{\wedge k,\wedge l}(u-v),Q^{\wedge k}\otimes Q^{\wedge l}\big]=0\tag{2.14}$$

for any even matrix $Q\in\mathrm{End}(\mathcal{V})$.

Let

$$R_{\wedge k,\wedge 1}(u)=u+\sum_{a,b=1}^{N}\sum_{i=1}^{k}(E_{ab}^{(i)})\big|_{\mathcal{V}^{\wedge k}}\otimes E_{ba}(-1)^{|b|}\in\mathrm{End}(\mathcal{V}^{\wedge k})\otimes\mathrm{End}(\mathcal{V}),$$

$$R_{\wedge 1,\wedge k}(u)=u+k-1+\sum_{a,b=1}^{N}\sum_{i=1}^{k}E_{ab}\otimes(E_{ba}^{(i)})\big|_{\mathcal{V}^{\wedge k}}(-1)^{|b|}\in\mathrm{End}(\mathcal{V})\otimes\mathrm{End}(\mathcal{V}^{\wedge k}).$$

Lemma 2.3. *We have*

$$R^{\wedge k,\wedge 1}(u)=R_{\wedge k,\wedge 1}(u)\prod_{i=1}^{k-1}(u-i),\qquad R^{\wedge 1,\wedge k}(u)=R_{\wedge 1,\wedge k}(u)\prod_{i=0}^{k-2}(u+i).\qquad\square$$

Corollary 2.4. *We have $R_{\wedge k,\wedge 1}(u)\big(R_{\wedge 1,\wedge k}(-u)\big)^{(21)}=(u+1)(k-u)$.*

Proof. This follows from the inversion relation of R-matrix, $R(u)R^{(21)}(-u)=1-u^2$. $\square$

Consider the series $T^{(k,k+1)}(u)\cdots T^{(1,k+1)}(u-k+1)$ with coefficients in $\mathrm{End}(\mathcal{V}^{\otimes k})\otimes \mathrm{Y}(\mathfrak{gl}_{m|n})$. By (2.5) and Lemma 2.2, we have

$$\mathbb{A}_{\{k\}}^{(1\dots k)}T^{(1,k+1)}(u-k+1)\cdots T^{(k,k+1)}(u)=T^{(k,k+1)}(u)\cdots T^{(1,k+1)}(u-k+1)\mathbb{A}_{\{k\}}^{(1\dots k)}.\tag{2.15}$$

Hence the space $\mathcal{V}^{\wedge k}$ is invariant under all coefficients of the series $T^{(k,k+1)}(u)\cdots T^{(1,k+1)}(u-k+1)$. Denote $T^{\wedge k}(u)$ the restriction of this series to $\mathrm{End}(\mathcal{V}^{\wedge k})\otimes \mathrm{Y}(\mathfrak{gl}_{m|n})$:

$$T^{\wedge k}(u) = T^{(k,k+1)}(u)\cdots T^{(1,k+1)}(u-k+1)\big|^{(1\ldots k)}_{\mathcal{V}^{\wedge k}}. \tag{2.16}$$

Note that, $T^{\wedge 1}(u) = T(u)$. Moreover, it follows from (2.5) and (2.13) that

$$\begin{aligned}\big(R^{\wedge k,\wedge l}(u-v)\big)^{(12)}\big(T^{\wedge k}(u)\big)^{(13)}\big(T^{\wedge l}(v)\big)^{(23)}&\\ = \big(T^{\wedge l}(v)\big)^{(23)}\big(T^{\wedge k}(u)\big)^{(13)}\big(R^{\wedge k,\wedge l}(u-v)\big)^{(12)}&.\end{aligned} \tag{2.17}$$

Clearly, by (2.8), we have

$$\begin{aligned}(\mathrm{id}\otimes\Delta)\big(T^{\wedge k}(u)\big) &= \big(T^{\wedge k}(u)\big)^{(13)}\big(T^{\wedge k}(u)\big)^{(12)},\\ (\mathrm{id}\otimes\widetilde{\Delta})\big(T^{\wedge k}(u)\big) &= \big(T^{\wedge k}(u)\big)^{(12)}\big(T^{\wedge k}(u)\big)^{(13)}.\end{aligned} \tag{2.18}$$

3. XXX SPIN CHAINS

3.1. **Higher transfer matrices.** For any even matrix $Q\in\mathrm{End}(\mathcal{V})$, define the series

$$\mathcal{T}_{k,Q}(u) = (\mathrm{str}_{\mathcal{V}^{\wedge k}}\otimes\mathrm{id})\big(Q^{\wedge k}T^{\wedge k}(u)\big), \qquad k\in\mathbb{Z}_{>0}, \tag{3.1}$$

with coefficients in $\mathrm{Y}(\mathfrak{gl}_{m|n})$. We call these series *transfer matrices*. By convention, we also set $\mathcal{T}_{0,Q}(u)=1$. Note that transfer matrices are even.

Lemma 3.1. *Transfer matrices satisfy the following properties.*

1. *Transfer matrices commute,* $[\mathcal{T}_{k,Q}(u),\mathcal{T}_{l,Q}(v)]=0$.
2. *If Q is the identity matrix, then coefficients of transfer matrices commute with the subalgebra* $\mathrm{U}(\mathfrak{gl}_{m|n})$ *in* $\mathrm{Y}(\mathfrak{gl}_{m|n})$.
3. *If Q is a diagonal matrix with pairwise distinct diagonal elements, then the subalgebra (the Bethe subalgebra) of* $\mathrm{Y}(\mathfrak{gl}_{m|n})$ *generated by all coefficients of all transfer matrices contains* $\mathrm{U}(\mathfrak{h})$*, where $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{gl}_{m|n}$.* □

One can also define another family of transfer matrices associated to symmetrizers

$$\mathfrak{T}_{k,Q}(u) = (\mathrm{str}_{\mathcal{V}^{\otimes k}}\otimes\mathrm{id})\big(\mathbb{H}^{(1\ldots k)}_{\{k\}}Q^{(1)}\cdots Q^{(k)}T^{(1,k+1)}(u)\cdots T^{(k,k+1)}(u-k+1)\big), \qquad k\in\mathbb{Z}_{>0}.$$

Again, we set $\mathfrak{T}_{0,Q}(u)=1$.

Transfer matrices can be compactly combined into a generating series using quantum Berezinian as follows, see [MR14] and cf. [Tal06].

We follow the convention of [MR14]. Let $\mathcal{A}$ be a superalgebra. Consider the operators of the form

$$\mathcal{K} = \sum_{a,b=1}^{N}(-1)^{|a||b|+|b|}E_{ab}\otimes K_{ab}\in\mathrm{End}(\mathcal{V})\otimes\mathcal{A}, \tag{3.2}$$

where K_{ab} are elements of $\mathcal{A}$ of parity $|a|+|b|$. We say that $\mathcal{K}$ is a *Manin matrix* if

$$[K_{ab},K_{cd}] = (-1)^{|a||b|+|a||c|+|b||c|}[K_{cb},K_{ad}]$$

for all $1\leqslant a,b,c,d\leqslant N$.

If $\mathcal{K}$ is invertible and has the form

$$\mathcal{K}^{-1} = \sum_{a,b=1}^{N} (-1)^{|a||b|+|b|} E_{ab} \otimes K'_{ab} \in \mathrm{End}(\mathcal{V}) \otimes \mathcal{A},$$

then we define the (quantum) *Berezinian* of $\mathcal{K}$ by

$$\mathrm{Ber}(\mathcal{K}) = \sum_{\sigma \in S_m} \mathrm{sign}(\sigma) K_{\sigma(1)1} \cdots K_{\sigma(m)m} \cdot \sum_{\tilde\sigma \in S_n} \mathrm{sign}(\sigma) K'_{m+1,m+\tilde\sigma(1)} \cdots K'_{m+n,m+\tilde\sigma(n)}. \tag{3.3}$$

Let $e^{-\partial_u}$ be the difference operator, $(e^{-\partial_u} f)(u) = f(u-1)$ for any function f in u. Let $\mathcal{A}$ be the superalgebra $\mathrm{Y}(\mathfrak{gl}_{m|n})[[u^{-1}, \partial_u]]$, where ∂_u is even. Here u and ∂_u satisfy the relations

$$\partial_u \cdot u^{-s} = u^{-s}\partial_u - s u^{-s-1}, \quad s \in \mathbb{Z}_{>0}.$$

Consider the operator $Z^Q(x, \partial_u)$,

$$Z_Q(u, \partial_u) = T^\dagger(u) Q^\dagger e^{-\partial_u} \in \mathrm{End}(\mathcal{V}) \otimes \mathrm{Y}(\mathfrak{gl}_{m|n})[[u^{-1}, \partial_u]].$$

It follows from (2.5) or (2.6) that $Z_Q(u, \partial_u)$ is a Manin matrix, see e.g. [MR14, Remark 2.12] and cf. [CF08, Proposition 4]. Note that our generating series $T_{ij}(u)$ corresponds to $z_{ji}(u)$ in [MR14].

Define the rational difference operator $\mathcal{D}^Q(u, \partial_u)$,

$$\mathcal{D}_Q(u, \partial_u) = \mathrm{Ber}(1 - Z_Q(u, \partial_u)). \tag{3.4}$$

Applying the supertransposition to all copies of $\mathrm{End}(\mathcal{V})$ and using cyclic property of supertrace, see (2.3), it follows from [MR14, Theorem 2.13] that

$$\mathcal{D}_Q(u, \partial_u) = \sum_{k=0}^{\infty} (-1)^k \mathfrak{T}_{k,Q}(u) e^{-k\partial_u}, \quad \big(\mathcal{D}_Q(u, \partial_u)\big)^{-1} = \sum_{k=0}^{\infty} \mathfrak{T}_{k,Q}(u) e^{-k\partial_u}. \tag{3.5}$$

3.2. **Universal off-shell Bethe vectors.** In this section, we recall the supertrace formula of Bethe vectors and its properties from [MTV06, BR08, PRS17].

Let $\boldsymbol{\xi} = (\xi^1, \dots, \xi^{N-1})$ be a sequence of nonnegative integers. Set $\xi^{<a} = \xi^1 + \cdots + \xi^{a-1}$, $1 < a \leqslant N$. In particular, we set $|\boldsymbol{\xi}| = \xi^{<N}$. Consider a series in $|\boldsymbol{\xi}|$ variables

$$\boldsymbol{t} = (t^1_1, \dots, t^1_{\xi^1}, \dots, t^{N-1}_1, \dots, t^{N-1}_{\xi^{N-1}})$$

with coefficients in $\mathrm{Y}(\mathfrak{gl}_{m|n})$,

$$\begin{aligned}\widehat{\mathbb{B}}_{\boldsymbol{\xi}}(\boldsymbol{t}) = (\mathrm{str} \otimes \mathrm{id})\Big(& T^{(1,|\boldsymbol{\xi}|+1)}(t^1_1) \cdots T^{(|\boldsymbol{\xi}|,|\boldsymbol{\xi}|+1)}(t^{N-1}_{\xi^{N-1}}) \\ & \times \prod_{(a,i)<(b,j)}^{\longrightarrow} R^{(\xi^{<b}+j, \xi^{<a}+i)}(t^b_j - t^a_i) E_{21}^{\otimes \xi^1} \otimes \cdots \otimes E_{N,N-1}^{\otimes \xi^{N-1}} \otimes 1\Big),\end{aligned} \tag{3.6}$$

where the supertrace is taken over all factors and the pairs are ordered lexicographically, namely $(a,i) < (b,j)$ if $a < b$, or $a = b$ and $i < j$. Moreover, the product is taken over the set $\{(c,k) \mid 1 \leqslant c < N,\ 1 \leqslant k \leqslant \xi^c\}$. Note that Bethe vectors are obtained by applying $\widehat{\mathbb{B}}_{\boldsymbol{\xi}}(\boldsymbol{t})$ to pseudovaccum vectors (singular ℓ-weight vectors). Therefore, we call $\widehat{\mathbb{B}}_{\boldsymbol{\xi}}(\boldsymbol{t})$ (and its renormalizations) a *universal off-shell Bethe vector*.

This supertrace formula is slightly different from the one in [BR08]. But it turns out that they only differ by a scalar function in $\boldsymbol{t}$, see [PRS17, Propositions 3.2 & 3.3].

It is clear from the Yang-Baxter equation and the equality

$$R(u-v)E_{ab}\otimes E_{ab}=(u-v+(-1)^{|a|})E_{ab}\otimes E_{ab}$$

that $\widehat{\mathbb{B}}_{\boldsymbol{\xi}}(\boldsymbol{t})$ is divisible by

$$\prod_{a=1}^{N-1}\prod_{1\leqslant i<j\leqslant \xi^a}(t_j^a-t_i^a+(-1)^{|a+1|}) \tag{3.7}$$

in $\mathrm{Y}(\mathfrak{gl}_{m|n})[t_1^1,\dots,t^{N-1}_{\xi^{N-1}}][[(t_1^1)^{-1},\dots,(t^{N-1}_{\xi^{N-1}})^{-1}]]$.

Recall the canonical projection $\theta:\mathrm{Y}(\mathfrak{gl}_{m|n})\to\mathrm{Y}(\mathfrak{gl}_{m|n})/\mathrm{Y}_+(\mathfrak{gl}_{m|n})$.

Lemma 3.2. *The series $\theta(\widehat{\mathbb{B}}_{\boldsymbol{\xi}}(\boldsymbol{t}))$ is divisible by*

$$\prod_{a=1}^{N-2}\prod_{b=a+2}^{N-1}\prod_{i=1}^{\xi^a}\prod_{j=1}^{\xi^b}(t_j^b-t_i^a)$$

in $(\mathrm{Y}(\mathfrak{gl}_{m|n})/\mathrm{Y}_+(\mathfrak{gl}_{m|n}))[t_1^1,\dots,t^{N-1}_{\xi^{N-1}}][[(t_1^1)^{-1},\dots,(t^{N-1}_{\xi^{N-1}})^{-1}]]$. □

The lemma will be proved in Section 4.1 after Proposition 4.3 where we recall the recursion for the Bethe vector.

Set

$$\begin{aligned}\mathbb{B}_{\boldsymbol{\xi}}(\boldsymbol{t})&=\widehat{\mathbb{B}}_{\boldsymbol{\xi}}(\boldsymbol{t})\prod_{a=1}^{N-1}\prod_{1\leqslant i<j\leqslant \xi^a}\frac{1}{t_j^a-t_i^a+(-1)^{|a+1|}}\prod_{1\leqslant a<b<N}\prod_{i=1}^{\xi^a}\prod_{j=1}^{\xi^b}\frac{1}{t_j^b-t_i^a},\\ \overline{\mathbb{B}}_{\boldsymbol{\xi}}(\boldsymbol{t})&=\mathbb{B}_{\boldsymbol{\xi}}(\boldsymbol{t})\prod_{1\leqslant i<j\leqslant \xi^m}\frac{1}{t_j^m-t_i^m-(-1)^{|m+1|}},\end{aligned} \tag{3.8}$$

see [PRS17, Equation (3.1) & Proposition 3.3]. Note that $\overline{\mathbb{B}}_{\boldsymbol{\xi}}(\boldsymbol{t})$ corresponds to the Bethe vector used in [PRS17] which is symmetric in variables t_i^a with the same superscript a for all $1\leqslant a<N$, see [PRS17, Proposition 3.2]. Examples of $\overline{\mathbb{B}}_{\boldsymbol{\xi}}(\boldsymbol{t})$ for small N can be found in [PRS17, Section 3.1].

Here we shall mainly use $\mathbb{B}_{\boldsymbol{\xi}}(\boldsymbol{t})$ with $\boldsymbol{t}$ satisfying

$$t_j^m-t_i^m-(-1)^{|m+1|}\neq 0,\quad 1\leqslant i<j\leqslant \xi^m, \tag{3.9}$$

due to the equality $T_{m+1,m}(u)T_{m+1,m}(u-1)=0$. Note that equation (3.9) always holds after reordering t_i^m e.g. in increasing order with respect to the real parts. Moreover, $\mathbb{B}_{\boldsymbol{\xi}}(\boldsymbol{t})$ is symmetric in variables t_i^a with the same superscript a for all $1\leqslant a<N$ except when $a=m$ (the corresponding simple root is odd).

In general, $\mathbb{B}_{\boldsymbol{\xi}}(\boldsymbol{t})$ is a sum of the products

$$T_{a_1,b_1}(t_1^1)\cdots T_{a_{|\boldsymbol{\xi}|},b_{|\boldsymbol{\xi}|}}(t^{N-1}_{\xi^{N-1}})p(\boldsymbol{t})\prod_{1\leqslant a<b<N}\prod_{i=1}^{\xi^a}\prod_{j=1}^{\xi^b}\frac{1}{t_j^b-t_i^a} \tag{3.10}$$

with various $a_1,\dots,a_{|\boldsymbol{\xi}|},b_1,\dots,b_{|\boldsymbol{\xi}|}$ from $\{1,\dots,N\}$ and polynomials $p(\boldsymbol{t})$.

3.3. **Main technical result.** We use the same notation as in Section 3.2. Following e.g. [HMVY19], introduce a sequence of polynomials $\boldsymbol{y} = (y_1, \dots, y_{N-1})$ associated to $\boldsymbol{t}$ and $\boldsymbol{\xi}$,

$$y_a(u) = \prod_{i=1}^{\xi^a} (u - t_i^a). \tag{3.11}$$

By convention $y_0 = y_N = 1$. We also set $\kappa_a = 1$ for $1 \leqslant a \leqslant m$ and $\kappa_b = -1$ for $m < b \leqslant N$.

From now on, we assume that $Q = \sum_{a=1}^N Q_a E_{aa}$ is diagonal. Define the series

$$\begin{aligned} \mathcal{J}^{a,i}_{\boldsymbol{\xi},Q}(\boldsymbol{t}) = \kappa_a Q_a T_{aa}(t_i^a) y_{a-1}(t_i^a + \kappa_a) y_a(t_i^a - \kappa_a) y_{a+1}(t_i^a) \\ + \kappa_{a+1} Q_{a+1} T_{a+1,a+1}(t_i^a) y_{a-1}(t_i^a) y_a(t_i^a + \kappa_{a+1}) y_{a+1}(t_i^a - \kappa_{a+1}) \end{aligned} \tag{3.12}$$

for $1 \leqslant a < N$, $1 \leqslant i \leqslant \xi^a$.

Given the data: integers $a_1, \dots, a_{|\boldsymbol{\xi}|+k-1}$, $b_1, \dots, b_{|\boldsymbol{\xi}|+k-1}$, $c \in \{1, \dots, N\}$ and $i \in \{1, \dots, \xi^c\}$, a sequence $s_1, \dots, s_{|\boldsymbol{\xi}|+k-1}$ which is a permutation of the sequence $u, \dots, u-k+1, t_1, \dots, \widehat{t_i^c}, \dots, t^{N-1}_{\xi^{N-1}}$, where the hat means that the corresponding variable t_i^c is skipped, and a polynomial $p(u;\boldsymbol{t})$, consider

$$\begin{aligned} T_{a_1,b_1}(s_1) \cdots T_{a_{|\boldsymbol{\xi}|+k-1}, b_{|\boldsymbol{\xi}|+k-1}}(s_{|\boldsymbol{\xi}|+k-1}) \mathcal{J}^{c,i}_{\boldsymbol{\xi},Q}(\boldsymbol{t}) p(u;\boldsymbol{t}) \\ \times \prod_{a=1}^{N-1} \prod_{j=1}^{\xi^a} \Big(\frac{1}{u - t_j^a} \prod_{1 \leqslant j < l \leqslant \xi^a} \frac{1}{t_j^a - t_l^a} \Big) \prod_{a=1}^{N-2} \prod_{j=1}^{\xi^a} \prod_{l=1}^{\xi^{a+1}} \frac{1}{(t_l^{a+1} - t_j^a)^2}. \end{aligned} \tag{3.13}$$

Here the factors $(u - t_j^a)^{-1}$ are considered as power series in u^{-1}. Denote by $I_{\boldsymbol{\xi},\boldsymbol{t},k,Q}$ the $\mathbb{C}$-span of all products (3.13) with all possible data. We also denote by $I_{\boldsymbol{\xi},\boldsymbol{t},Q}$ the sum of $I_{\boldsymbol{\xi},\boldsymbol{t},k,Q}$ for $k \in \mathbb{Z}_{>0}$.

For $1 \leqslant a \leqslant N$, define the series

$$\mathcal{X}^a_{\boldsymbol{\xi},Q}(u;\boldsymbol{t}) = Q_a T_{aa}(u) \frac{y_{a-1}(u + \kappa_a) y_a(u - \kappa_a)}{y_{a-1}(u) y_a(u)}, \tag{3.14}$$

which are regarded as power series in u^{-1} with coefficients in $\mathrm{Y}(\mathfrak{gl}_{m|n})[t_1^1, \dots, t^{N-1}_{\xi^{N-1}}]$.

Theorem 3.3. *Let Q be a diagonal matrix. Then we have*

$$\mathcal{T}_{k,Q}(u) \mathbb{B}_{\boldsymbol{\xi}}(\boldsymbol{t}) \simeq \mathbb{B}_{\boldsymbol{\xi}}(\boldsymbol{t}) \sum_{\boldsymbol{a}} \prod_{r=1}^{k} \kappa_{a_r} \mathcal{X}^{a_r}_{\boldsymbol{\xi},Q}(u - r + 1; \boldsymbol{t}) + \mathcal{U}_{\boldsymbol{\xi},k,Q}(u;\boldsymbol{t}),$$

where the sum is taken over all k-tuples $\boldsymbol{a} = (1 \leqslant a_1 < \cdots < a_b < m+1 \leqslant a_{b+1} \leqslant \cdots \leqslant a_k \leqslant N)$ for various $0 \leqslant b \leqslant k$ and $\mathcal{U}_{\boldsymbol{\xi},k,Q}(u;\boldsymbol{t})$ is in $I_{\boldsymbol{\xi},\boldsymbol{t},k,Q}$.

The theorem is proved in Section 4.3.

Note that, due to Lemma 2.1, the order of $\mathcal{X}^{a_r}_{\boldsymbol{\xi},Q}(u - r + 1; \boldsymbol{t})$ in Theorem 3.3 is irrelevant.

Corollary 3.4. *Let Q be a diagonal matrix. Then we have*

$$\mathcal{D}_Q(u, \partial_u) \mathbb{B}_{\boldsymbol{\xi}}(\boldsymbol{t}) \simeq \mathbb{B}_{\boldsymbol{\xi}}(\boldsymbol{t}) \overrightarrow{\prod_{1 \leqslant a \leqslant N}} \big(1 - \mathcal{X}^a_{\boldsymbol{\xi},Q}(u;\boldsymbol{t}) e^{-\partial_u}\big)^{\kappa_a} + \mathcal{U}_{\boldsymbol{\xi},Q}(u;\boldsymbol{t}),$$

where $\mathcal{U}_{\boldsymbol{\xi},Q}(u;\boldsymbol{t})$ belongs to $I_{\boldsymbol{\xi},\boldsymbol{t},Q}$ and $\mathcal{D}_Q(u, \partial_u)$is defined in (3.4).

Proof. This follows from direct computations using (3.5) and Theorem 3.3. ☐

3.4. **Main results for XXX spin chains.** In this section, we shall obtain eigenvectors and eigenvalues of transfer matrices when the underlying Hilbert spaces are tensor products of evaluation modules of the super Yangian $\mathrm{Y}(\mathfrak{gl}_{m|n})$, proving [LM21b, Conjecture 5.15]. For more general highest weight representations of $\mathrm{Y}(\mathfrak{gl}_{m|n})$, [LM21b, Conjecture 5.15] is proved similarly for generic situation.

Let ℓ be a positive integer. Note that ℓ here has nothing to do with ℓ in ℓ-weight. Let $M_1,\dots,M_\ell$ be $\mathfrak{gl}_{m|n}$-modules, $\boldsymbol{z}=(z_1,\dots,z_\ell)$ a sequence of complex numbers. Consider the tensor product of evaluation $\mathrm{Y}(\mathfrak{gl}_{m|n})$-modules,

$$M(\boldsymbol{z}):=M_1(z_1)\otimes\cdots\otimes M_\ell(z_\ell).$$

Then, by (2.9) and (2.10), the operator

$$\mathcal{T}^M_{k,Q}(u;\boldsymbol{z})=\mathcal{T}_{k,Q}(u)\big|_{M(\boldsymbol{z})} \tag{3.15}$$

is a rational function in $u,\boldsymbol{z}$ with denominators $\prod_{i=1}^{\ell}\prod_{j=0}^{k-1}(u-j-z_i)$. Note that

$$\mathcal{T}^M_{k,Q}(u;\boldsymbol{z})=\mathrm{str}_{\mathcal{V}^{\wedge k}}Q^{\wedge k}+O(u^{-1}),\qquad u\to\infty.$$

We call the operators $\mathcal{T}^M_{k,Q}(u;\boldsymbol{z})$, $k\in\mathbb{Z}_{>0}$, the *transfer matrices of the XXX spin chain* on $M(\boldsymbol{z})$ associated with $\mathfrak{gl}_{m|n}$.

We are interested in the case when $M_1,\dots,M_\ell$ are highest weight $\mathfrak{gl}_{m|n}$-modules with highest weights $\Lambda_1,\dots,\Lambda_\ell$, where $\Lambda_i=(\Lambda_i^1,\dots,\Lambda_i^N)$, and highest weight vectors $v_1,\dots,v_\ell$, respectively. By convention, we set $\boldsymbol{\Lambda}=(\Lambda_1,\dots,\Lambda_\ell)$.

In this case, the vector $v^+=v_1\otimes\cdots\otimes v_\ell$ is a singular ℓ-weight vector of $\mathrm{Y}(\mathfrak{gl}_{m|n})$ in $M(\boldsymbol{z})$ whose ℓ-weight is given as follows,

$$T_{aa}(u)v^+=v^+\prod_{i=1}^{\ell}\frac{u-z_i+\kappa_a\Lambda_i^a}{u-z_i},\qquad 1\leqslant a\leqslant N. \tag{3.16}$$

Recall $\mathbb{B}_{\boldsymbol{\xi}}(\boldsymbol{t})$ and the notations from Section 3.2. Apply $\mathbb{B}_{\boldsymbol{\xi}}(\boldsymbol{t})$ to v^+ and renormalize it so that the function

$$\mathbb{B}^{v^+}_{\boldsymbol{\xi}}(\boldsymbol{t};\boldsymbol{z})=\mathbb{B}_{\boldsymbol{\xi}}(\boldsymbol{t})v^+\prod_{a=1}^{N-1}\prod_{i=1}^{\xi^a}\prod_{j=1}^{\ell}(t_i^a-z_j)\prod_{a=1}^{N-2}\prod_{i=1}^{\xi^a}\prod_{j=1}^{\xi^{a+1}}(t_j^{a+1}-t_i^a) \tag{3.17}$$

is a polynomial in $\boldsymbol{t}$, $\boldsymbol{z}$, see Lemma 3.2 and (3.10). We call $\mathbb{B}^{v^+}_{\boldsymbol{\xi}}(\boldsymbol{t};\boldsymbol{z})$ the *off-shell Bethe vector* for the XXX spin chain on $M(\boldsymbol{z})$ associated with $\mathfrak{gl}_{m|n}$.

Recall that $Q=\sum_{a=1}^{N}Q_aE_{aa}$ is diagonal and let $\boldsymbol{y}$ be the sequence of polynomials associated to $\boldsymbol{t}$ and $\boldsymbol{\xi}$. Consider the system of algebraic equations

$$\begin{aligned}&-\kappa_aQ_a\prod_{j=1}^{\ell}(t_i^a-z_j+\kappa_a\Lambda_j^a)y_{a-1}(t_i^a+\kappa_a)y_a(t_i^a-\kappa_a)y_{a+1}(t_i^a)\\&=\kappa_{a+1}Q_{a+1}\prod_{j=1}^{\ell}(t_i^a-z_j+\kappa_{a+1}\Lambda_j^{a+1})y_{a-1}(t_i^a)y_a(t_i^a+\kappa_{a+1})y_{a+1}(t_i^a-\kappa_{a+1}),\end{aligned} \tag{3.18}$$

where $1\leqslant a<N$, $1\leqslant i\leqslant\xi^a$, $y_0=y_N=1$. We call (3.18) the *Bethe ansatz equation*. We say that a solution $\tilde{\boldsymbol{t}}=(\tilde t_1^1,\dots,\tilde t^{N-1}_{\xi^{N-1}})$ of the Bethe ansatz equation (3.18) is *off-diagonal* if $\tilde t_i^a\neq\tilde t_j^a$ for any

$1 \leqslant a < N$, $1 \leqslant i < j \leqslant \xi^a$ and $\tilde{t}_i^a \neq \tilde{t}_j^{a+1}$ for any $1 \leqslant a < N-1$, $1 \leqslant i \leqslant \xi^a$, $1 \leqslant j \leqslant \xi^{a+1}$ (aslo (3.9)).

When $\tilde{\boldsymbol{t}}$ is an off-diagonal solution of the Bethe ansatz equation (3.18), we say that the vector $\mathbb{B}^{v^+}_{\boldsymbol{\xi}}(\tilde{\boldsymbol{t}};\boldsymbol{z})$ is an *on-shell Bethe vector*.

For $1 \leqslant a \leqslant N$, define

$$\mathcal{X}^a_{\boldsymbol{\xi},Q}(u;\boldsymbol{t};\boldsymbol{z};\boldsymbol{\Lambda}) = Q_a \frac{y_{a-1}(u+\kappa_a)y_a(u-\kappa_a)}{y_{a-1}(u)y_a(u)} \prod_{i=1}^{\ell} \frac{u-z_i+\kappa_a\Lambda_i^a}{u-z_i}, \tag{3.19}$$

where $\boldsymbol{y} = (y_1, \dots, y_{N-1})$ is the sequence of polynomials associated to $\boldsymbol{t}$ and $\boldsymbol{\xi}$.

Theorem 3.5. *Let Q be a diagonal matrix. If $M_1, \dots, M_\ell$ are highest weight $\mathfrak{gl}_{m|n}$-modules with highest weights $\Lambda_1, \dots, \Lambda_\ell$ and $\tilde{\boldsymbol{t}}$ is an off-diagonal solution of the Bethe ansatz equation* (3.18)*, then we have*

$$\mathcal{T}_{k,Q}(u)\mathbb{B}^{v^+}_{\boldsymbol{\xi}}(\tilde{\boldsymbol{t}};\boldsymbol{z}) = \mathbb{B}^{v^+}_{\boldsymbol{\xi}}(\tilde{\boldsymbol{t}};\boldsymbol{z}) \sum_{\boldsymbol{a}} \prod_{r=1}^{k} \kappa_{a_r} \mathcal{X}^{a_r}_{\boldsymbol{\xi},Q}(u-r+1;\tilde{\boldsymbol{t}};\boldsymbol{z};\boldsymbol{\Lambda}),$$

where the sum is taken over all k-tuples $\boldsymbol{a} = (1 \leqslant a_1 < \cdots < a_b < m+1 \leqslant a_{b+1} \leqslant \cdots \leqslant a_k \leqslant N)$ for various $0 \leqslant b \leqslant k$.

The proof of the theorem is given in Section 4.3. The statement was shown in [MTV06, Theorem 5.4] for the general even case and conjectured in [LM21b, Conjecture 5.15]. The case of $m = n = 1$ was previously shown in [LM21a, Theorem 6.1]. It can be thought as the supersymmetric version of [FH15, Theorem 5.11] and [FJMM17, Theorem 7.5] for type A. When $k = 1$, the statement was obtained in [BR08].

Corollary 3.6. *Let Q be a diagonal matrix. If $M_1, \dots, M_\ell$ are highest weight $\mathfrak{gl}_{m|n}$-modules with highest weights $\Lambda_1, \dots, \Lambda_\ell$ and $\tilde{\boldsymbol{t}}$ is an off-diagonal solution of the Bethe ansatz equation* (3.18)*, then we have*

$$\mathcal{D}_Q(u,\partial_u)\mathbb{B}^{v^+}_{\boldsymbol{\xi}}(\tilde{\boldsymbol{t}};\boldsymbol{z}) = \mathbb{B}^{v^+}_{\boldsymbol{\xi}}(\tilde{\boldsymbol{t}};\boldsymbol{z}) \overrightarrow{\prod_{1\leqslant a\leqslant N}} \left(1 - \mathcal{X}^a_{\boldsymbol{\xi},Q}(u;\tilde{\boldsymbol{t}};\boldsymbol{z};\boldsymbol{\Lambda})e^{-\partial_u}\right)^{\kappa_a},$$

where $\mathcal{D}_Q(u,\partial_u)$is defined in (3.4).

Note that the rational difference operator

$$\mathcal{D}_Q(u,\partial_u;\tilde{\boldsymbol{t}};\boldsymbol{z};\boldsymbol{\Lambda}) := \overrightarrow{\prod_{1\leqslant a\leqslant N}} \left(1 - \mathcal{X}^a_{\boldsymbol{\xi},Q}(u;\tilde{\boldsymbol{t}};\boldsymbol{z};\boldsymbol{\Lambda})e^{-\partial_u}\right)^{\kappa_a} \tag{3.20}$$

was introduced in [HLM19, Equation (5.6)], cf. [Tsu97, Equations (2.13)].

Proposition 3.7. *Let Q be the identity matrix. If $M_1, \dots, M_\ell$ are highest weight $\mathfrak{gl}_{m|n}$-modules with highest weights $\Lambda_1, \dots, \Lambda_\ell$ and $\tilde{\boldsymbol{t}}$ is an off-diagonal solution of the Bethe ansatz equation* (3.18)*, then the on-shell Bethe vector $\mathbb{B}^{v^+}_{\boldsymbol{\xi}}(\tilde{\boldsymbol{t}};\boldsymbol{z})$ is a $\mathfrak{gl}_{m|n}$-singular vector in $M_1 \otimes \cdots \otimes M_\ell$ with weight*

$$\Big(\sum_{i=1}^{\ell} \Lambda_i^1 - \xi^1, \sum_{i=1}^{\ell} \Lambda_i^1 + \xi^1 - \xi^2, \dots, \sum_{i=1}^{\ell} \Lambda_i^N + \xi^{N-1}\Big).$$

The proof of the proposition is given in Section 4.3.

4. Proof of main results

We start with preparing a few statements which will be used in the proof.

4.1. **Recursion for the Bethe vectors.** Since we shall use nested algebraic Bethe ansatz, many notations will be used for both $\mathfrak{gl}_{m|n}$ and $\mathfrak{gl}_{m-1|n}$. To simplify the notation, we use $\mathcal{N}$ and $\mathcal{N}-1$ to distinguish notations for $\mathfrak{gl}_{m|n}$ and $\mathfrak{gl}_{m-1|n}$, respectively. We also use $\langle\mathcal{N}\rangle$ and $\langle\mathcal{N}-1\rangle$.

Set $\mathcal{W}=\mathbb{C}^{\mathcal{N}-1}=\mathbb{C}^{m-1|n}$. Let $\mathbf{w}_1,\dots,\mathbf{w}_{N-1}$ be the standard basis of $\mathcal{W}$ and $\mathbf{v}_1,\dots,\mathbf{v}_N$ of $\mathcal{V}=\mathbb{C}^{\mathcal{N}}=\mathbb{C}^{m|n}$. Identify $\mathcal{W}$ with the subspace of $\mathcal{V}$ via the embedding $\mathbf{w}_a\mapsto\mathbf{v}_{a+1}$, $1\leqslant a<N$.

Let $P^{\langle\mathcal{N}-1\rangle}\in\mathrm{End}(\mathcal{W}^{\otimes 2})$ be the graded flip operator and $R^{\langle\mathcal{N}-1\rangle}(u)=u+P^{\langle\mathcal{N}-1\rangle}$ be the rational R-matrix used to define the super Yangian $\mathrm{Y}(\mathfrak{gl}_{\mathcal{N}-1})$. The R-matrix $R(u)$ preserves the subspace $\mathcal{W}^{\otimes 2}\subset\mathcal{V}^{\otimes 2}$ and the restriction of $R(u)$ on $\mathcal{W}^{\otimes 2}$ coincides with $R^{\langle\mathcal{N}-1\rangle}(u)$. Recall that $\mathcal{W}(x)$ is the evaluation $\mathrm{Y}(\mathfrak{gl}_{\mathcal{N}-1})$-module with the corresponding homomorphism $\pi(x):\mathrm{Y}(\mathfrak{gl}_{\mathcal{N}-1})\to\mathrm{End}(\mathcal{W})$,

$$\pi(x):T^{\langle\mathcal{N}-1\rangle}(u)\mapsto(u-x)^{-1}R^{\langle\mathcal{N}-1\rangle}(u-x). \tag{4.1}$$

Define the embedding $\psi:\mathrm{Y}(\mathfrak{gl}_{\mathcal{N}-1})\hookrightarrow\mathrm{Y}(\mathfrak{gl}_{\mathcal{N}})$ by the rule $\psi(T^{\langle\mathcal{N}-1\rangle}_{ab}(u))=T_{a+1,b+1}(u)$, $1\leqslant a,b\leqslant N-1$. Note that $\psi(\mathrm{Y}_+(\mathfrak{gl}_{\mathcal{N}-1}))\subset\mathrm{Y}_+(\mathfrak{gl}_{\mathcal{N}})$.

Define a map $\psi(x_1,\dots,x_r):\mathrm{Y}(\mathfrak{gl}_{\mathcal{N}-1})\to\mathrm{Y}(\mathfrak{gl}_{\mathcal{N}})\otimes\mathrm{End}(\mathcal{W}^{\otimes r})$ by

$$\psi(x_1,\dots,x_r)=(\psi\otimes\pi(x_r)\otimes\cdots\otimes\pi(x_1))\circ(\widetilde{\Delta}^{\langle\mathcal{N}-1\rangle})^{(r)}, \tag{4.2}$$

where $(\widetilde{\Delta}^{\langle\mathcal{N}-1\rangle})^{(r)}:\mathrm{Y}(\mathfrak{gl}_{\mathcal{N}})\to\mathrm{Y}(\mathfrak{gl}_{\mathcal{N}})^{\otimes(r+1)}$ is the multiple opposite coproduct. Note that here we use the opposite coproduct $\widetilde{\Delta}$ which is consistent with that in [BR08].

Define a map $\widetilde{\psi}:\mathrm{Y}(\mathfrak{gl}_{\mathcal{N}-1})\to\mathrm{Y}(\mathfrak{gl}_{\mathcal{N}})\otimes\mathcal{W}^{\otimes r}$ by

$$\widetilde{\psi}(x_1,\dots,x_r)=\psi(x_1,\dots,x_r)(1\otimes\mathbf{w}_1^{\otimes r}).$$

The following lemmas are straightforward.

Lemma 4.1. *We have* $\widetilde{\psi}(x_1,\dots,x_r)(\mathrm{Y}_+(\mathfrak{gl}_{\mathcal{N}-1}))\subset\mathrm{Y}_+(\mathfrak{gl}_{\mathcal{N}})\otimes\mathcal{W}^{\otimes r}$.

Similarly, define the embedding $\phi:\mathrm{Y}(\mathfrak{gl}_{\mathcal{N}-2})\hookrightarrow\mathrm{Y}(\mathfrak{gl}_{\mathcal{N}-1})$ by the rule $\phi(T^{\langle\mathcal{N}-2\rangle}_{ab}(u))=T^{\langle\mathcal{N}-1\rangle}_{a+1,b+1}(u)$, $1\leqslant a,b\leqslant N-2$. Recall the canonical projection $\theta:\mathrm{Y}(\mathfrak{gl}_{\mathcal{N}})\twoheadrightarrow\mathrm{Y}(\mathfrak{gl}_{\mathcal{N}})/\mathrm{Y}_+(\mathfrak{gl}_{\mathcal{N}})$.

Lemma 4.2. *We have* $(\theta\otimes\mathrm{id}^{\otimes r})\widetilde{\psi}(x_1,\dots,x_r)\circ\phi=(\theta\circ\psi\circ\phi)\otimes\mathbf{w}_1^{\otimes r}$.

Set $\bar{\boldsymbol{\xi}}=(\xi^2,\dots,\xi^{N-1})$ and $\bar{\boldsymbol{t}}=(t_1^2,\dots,t^2_{\xi^2};\dots;t_1^{N-1},\dots,t^{N-1}_{\xi^{N-1}})$.

Proposition 4.3 ([BR08, Eq. (5.1)])**.** *We have*

$$\mathbb{B}_{\boldsymbol{\xi}}(\boldsymbol{t})=B^{(1)}(t_1^1)\cdots B^{(\xi^1)}(t^1_{\xi^1})\widetilde{\psi}(t_1^1,\dots,t^1_{\xi^1})\big(\mathbb{B}^{\langle\mathcal{N}-1\rangle}_{\bar{\boldsymbol{\xi}}}(\bar{\boldsymbol{t}})\big)$$

where $B(u)=(T_{12}(u),\dots,T_{1N}(u))=\sum_{a=1}^{N-1}E_{1,a+1}\otimes T_{1,a+1}(u)$ *and its coefficients are treated as elements in* $\mathrm{Hom}(\mathcal{W},\mathbb{C})\otimes\mathrm{Y}(\mathfrak{gl}_{m|n})$.

Proof of Lemma 3.2. By Lemma 4.2, Proposition 4.3, (4.1) and (4.2), the denominator of $\theta(\mathbb{B}_{\boldsymbol{\xi}}(\boldsymbol{t}))$ is at most

$$\prod_{a=1}^{N-2}\prod_{i=1}^{\xi^a}\prod_{j=1}^{\xi^{a+1}}(t_j^{a+1}-t_i^a).$$

Then the lemma follows from (3.8). $\square$

4.2. **Correspondence between $\mathrm{Y}(\mathfrak{gl}_{m|n})$ and $\mathrm{Y}(\mathfrak{gl}_{n|m})$.** For $1 \leqslant a \leqslant N$, set $a' = N + 1 - a$. In the following, we use a' and b' for the indices corresponding to the super Yangian $\mathrm{Y}(\mathfrak{gl}_{n|m})$. Moreover, their parities should be the parities inherited from $\mathrm{Y}(\mathfrak{gl}_{n|m})$. We have the isomorphism

$$\varpi : \mathrm{Y}(\mathfrak{gl}_{m|n}) \to \mathrm{Y}(\mathfrak{gl}_{n|m}), \quad T_{ab}(u) \to \widetilde{T}_{b'a'}(u)(-1)^{|a'||b'|+|b'|}.$$

Here and below, we shall use tilde to indicate the notations corresponding to $\mathrm{Y}(\mathfrak{gl}_{n|m})$. Note that ϖ maps $\mathrm{Y}_+(\mathfrak{gl}_{m|n})$ to $\mathrm{Y}_+(\mathfrak{gl}_{n|m})$.

We now describe the image of the rational difference operator (3.5) under the isomorphism ϖ. Recall the transfer matrices associated to symmetrizers,

$$\mathfrak{T}_{k,Q}(u) = (\mathrm{str}_{\mathcal{V}^{\otimes k}} \otimes \mathrm{id})\big(\mathbb{H}^{(1\dots k)}_{\{k\}} Q^{(1)} \cdots Q^{(k)} T^{(1,k+1)}(u) \cdots T^{(k,k+1)}(u-k+1)\big),$$

for $k \in \mathbb{Z}_{>0}$.

Set $\widetilde{\mathcal{V}} = \mathbb{C}^{n|m}$. We can identify $\mathcal{V}$ and $\widetilde{\mathcal{V}}$ by identifying $\mathbf{v}_a$ with $\widetilde{\mathbf{v}}_{a'}$, $1 \leqslant a \leqslant N$. Note that the parities are changed under this identification and the operator E_{ab} is identified with the operator $E_{a'b'}$. Moreover,

$$(\mathrm{str}_{\mathcal{V}} \otimes \mathrm{id})(E_{ab}) = (-1)^{|a|}\delta_{ab} = -(-1)^{|a'|}\delta_{ab} = -(\mathrm{str}_{\widetilde{\mathcal{V}}} \otimes \mathrm{id})(E_{a'b'}).$$

Example 4.4. We have the following identification

$$Q = \sum_{a=1}^{N} Q_a E_{aa} \in \mathrm{End}(\mathcal{V}) \longrightarrow \mathsf{Q} = \sum_{a=1}^{N} Q_a E_{a'a'} \in \mathrm{End}(\widetilde{\mathcal{V}}).$$

Due to $(-1)^{|a|} = -(-1)^{|a'|}$ (i.e. $\kappa_a = -\tilde{\kappa}_{a'}$), the R-matrix $R(u) \in \mathrm{End}(\mathcal{V}^{\otimes 2})$ is identified with $-\widetilde{R}(-u) \in \mathrm{End}(\widetilde{\mathcal{V}}^{\otimes 2})$. In particular, by Lemma 2.2, the action of $\mathbb{H}_{\{k\}}$ on $\mathcal{V}^{\otimes k}$ is the same as that of $\widetilde{\mathbb{A}}_{\{k\}}$ on $\widetilde{\mathcal{V}}^{\otimes k}$.

Define the matrix

$$\widetilde{\mathsf{T}}(u) := \big(\widetilde{T}(u)\big)^{\dagger} = \sum_{a',b'=1}^{N} E_{a'b'} \otimes \widetilde{T}_{b'a'}(u)(-1)^{|a'||b'|+|b'|} \in \mathrm{End}(\tilde{\mathcal{V}}) \otimes \mathrm{Y}(\mathfrak{gl}_{0|n})[[u^{-1}]].$$

Observe that

$$\varpi(T(u)) = \widetilde{\mathsf{T}}(u). \tag{4.3}$$

Lemma 4.5. *We have* $\varpi(\mathfrak{T}_{k,Q}(u)) = (-1)^k \widetilde{\mathfrak{I}}_{k,\mathsf{Q}}(u)$ *and* $\varpi(\mathfrak{I}_{k,Q}(u)) = (-1)^k \widetilde{\mathfrak{T}}_{k,\mathsf{Q}}(u)$.

Proof. By $\mathsf{Q}^{\dagger} = \mathsf{Q}$, $\big(\widetilde{\mathbb{A}}_{\{k\}}\big)^{\dagger} = \widetilde{\mathbb{A}}_{\{k\}}$, and the cyclicity of supertrace, we have

$$\begin{aligned}
\varpi(\mathfrak{T}_{k,Q}(u)) &= (-1)^k (\mathrm{str}_{\widetilde{\mathcal{V}}^{\otimes k}} \otimes \mathrm{id})\big(\widetilde{\mathbb{A}}^{(1\dots k)}_{\{k\}} \mathsf{Q}^{(1)} \cdots \mathsf{Q}^{(k)} \widetilde{\mathsf{T}}^{(1,k+1)}(u) \cdots \widetilde{\mathsf{T}}^{(k,k+1)}(u-k+1)\big) \\
&= (-1)^k (\mathrm{str}_{\widetilde{\mathcal{V}}^{\otimes k}} \otimes \mathrm{id})\big(\widetilde{T}^{(1,k+1)}(u) \cdots \widetilde{T}^{(k,k+1)}(u-k+1) \mathsf{Q}^{(1)} \cdots \mathsf{Q}^{(k)} \widetilde{\mathbb{A}}^{(1\dots k)}_{\{k\}}\big) \\
&= (-1)^k (\mathrm{str}_{\widetilde{\mathcal{V}}^{\otimes k}} \otimes \mathrm{id})\big(\widetilde{T}^{(1,k+1)}(u) \cdots \widetilde{T}^{(k,k+1)}(u-k+1) \widetilde{\mathbb{A}}^{(1\dots k)}_{\{k\}} \mathsf{Q}^{(1)} \cdots \mathsf{Q}^{(k)}\big) \\
&= (-1)^k (\mathrm{str}_{\widetilde{\mathcal{V}}^{\otimes k}} \otimes \mathrm{id})\big(\mathsf{Q}^{(1)} \cdots \mathsf{Q}^{(k)} \widetilde{T}^{(1,k+1)}(u) \cdots \widetilde{T}^{(k,k+1)}(u-k+1) \widetilde{\mathbb{A}}^{(1\dots k)}_{\{k\}}\big) \\
&= (-1)^k \widetilde{\mathfrak{I}}_{k,\mathsf{Q}}(u).
\end{aligned}$$

The other one is similar. ∎

In particular, by (3.5), we obtain

Corollary 4.6. *We have* $\varpi(\mathcal{D}_Q(u,\partial_u)) = \big(\widetilde{\mathcal{D}}_{\mathsf{Q}}(u,\partial_u)\big)^{-1}$.

We then consider the image of the universal off-shell Bethe vectors (3.6), (3.8) under the isomorphism ϖ. Let $\boldsymbol{\xi} = (\xi^1,\dots,\xi^{N-1})$ be a sequence of nonnegative integers,

$$\boldsymbol{t} = (t_1^1,\dots,t_{\xi^1}^1;\dots;t_1^{N-1},\dots,t_{\xi^{N-1}}^{N-1})$$

a sequence of variables. Set $\bar{\boldsymbol{\xi}} = (\xi^{N-1},\dots,\xi^1)$ and $\bar{\boldsymbol{t}} = (t_{\xi^{N-1}}^{N-1},\dots,t_1^{N-1};\dots;t_{\xi^1}^1,\dots,t_1^1)$.

By the fact that $(R(u))^\dagger = R(u)$ and Yang-Baxter equation (2.4), we have

$$\Big(\prod_{(a,i)<(b,j)}^{\longrightarrow} R^{(\xi^{<b}+j,\xi^{<a}+i)}(t_j^b-t_i^a)\Big)^\dagger = \prod_{(a,i)<(b,j)}^{\longrightarrow} R^{(\xi^{<b}+j,\xi^{<a}+i)}(t_j^b-t_i^a), \tag{4.4}$$

where $\dagger$ is taken over all factors.

Lemma 4.7 (cf. [PRS17, Lemma 5.1])**.** *The image of* $\mathbb{B}_{\boldsymbol{\xi}}(\boldsymbol{t})$ *under the isomorphism* ϖ *equals to* $\widetilde{\mathbb{B}}_{\bar{\boldsymbol{\xi}}}(\bar{\boldsymbol{t}})$ *up to sign.*

Proof. Instead of working on $\mathbb{B}_{\boldsymbol{\xi}}(\boldsymbol{t})$, we apply ϖ to $\widehat{\mathbb{B}}_{\boldsymbol{\xi}}(\boldsymbol{t})$. Note that $R(u)\in \mathrm{End}(\mathcal{V}^{\otimes 2})$ corresponds to $\widetilde{-R}(-u)\in\mathrm{End}(\widetilde{\mathcal{V}}^{\otimes 2})$. In the following, we use the symbol $\propto$ to denote the proportionality up to signs. Then we have

$$\begin{aligned}
\varpi(\widehat{\mathbb{B}}_{\boldsymbol{\xi}}(\boldsymbol{t})) &\overset{(4.3)}{\propto} (\mathrm{str}\otimes\mathrm{id})\Big(\widetilde{\mathsf{T}}^{(1,|\boldsymbol{\xi}|+1)}(t_1^1)\cdots\widetilde{\mathsf{T}}^{(|\boldsymbol{\xi}|,|\boldsymbol{\xi}|+1)}(t_{\xi^{N-1}}^{N-1})\\
&\qquad\times\prod_{(a,i)<(b,j)}^{\longrightarrow}\widetilde{R}^{(\xi^{<b}+j,\xi^{<a}+i)}(t_i^a-t_j^b)E_{2'1'}^{\otimes\xi^1}\otimes\cdots\otimes E_{N',(N-1)'}^{\otimes\xi^{N-1}}\otimes 1\Big)\\
&\overset{(2.3)}{\propto} (\mathrm{str}\otimes\mathrm{id})\Big(\big(E_{1'2'}^{\otimes\xi^1}\otimes\cdots\otimes E_{(N-1)',N'}^{\otimes\xi^{N-1}}\otimes 1\big)\\
&\qquad\times\prod_{(a,i)<(b,j)}^{\longrightarrow}\widetilde{R}^{(\xi^{<b}+j,\xi^{<a}+i)}(t_i^a-t_j^b)\widetilde{T}^{(1,|\boldsymbol{\xi}|+1)}(t_1^1)\cdots\widetilde{T}^{(|\boldsymbol{\xi}|,|\boldsymbol{\xi}|+1)}(t_{\xi^{N-1}}^{N-1})\Big),\\
&\overset{(2.2)}{\propto} (\mathrm{str}\otimes\mathrm{id})\Big(\prod_{(a,i)<(b,j)}^{\longrightarrow}\widetilde{R}^{(\xi^{<b}+j,\xi^{<a}+i)}(t_i^a-t_j^b)\widetilde{T}^{(1,|\boldsymbol{\xi}|+1)}(t_1^1)\cdots\\
&\qquad\times\widetilde{T}^{(|\boldsymbol{\xi}|,|\boldsymbol{\xi}|+1)}(t_{\xi^{N-1}}^{N-1})E_{1'2'}^{\otimes\xi^1}\otimes\cdots\otimes E_{(N-1)',N'}^{\otimes\xi^{N-1}}\otimes 1\Big),\\
&\overset{(2.5)}{\propto} (\mathrm{str}\otimes\mathrm{id})\Big(\widetilde{T}^{(|\boldsymbol{\xi}|,|\boldsymbol{\xi}|+1)}(t_{\xi^{N-1}}^{N-1})\cdots\widetilde{T}^{(1,|\boldsymbol{\xi}|+1)}(t_1^1)\\
&\qquad\times\prod_{(a,i)<(b,j)}^{\longrightarrow}\widetilde{R}^{(\xi^{<b}+j,\xi^{<a}+i)}(t_i^a-t_j^b)E_{1'2'}^{\otimes\xi^1}\otimes\cdots\otimes E_{(N-1)',N'}^{\otimes\xi^{N-1}}\otimes 1\Big),\\
&\propto (-1)^{|\boldsymbol{\xi}|}(\mathrm{str}\otimes\mathrm{id})\Big(\widetilde{T}^{(1,|\boldsymbol{\xi}|+1)}(t_{\xi^{N-1}}^{N-1})\cdots\widetilde{T}^{(|\boldsymbol{\xi}|,|\boldsymbol{\xi}|+1)}(t_1^1)\\
&\qquad\times\prod_{(a,i)<(b,j)}^{\longrightarrow}\widetilde{R}^{(|\boldsymbol{\xi}|+1-\xi^{<b}-j,|\boldsymbol{\xi}|+1-\xi^{<a}-i)}(t_i^a-t_j^b)E_{(N-1)',N'}^{\otimes\xi^{N-1}}\otimes\cdots\otimes E_{1'2'}^{\otimes\xi^1}\otimes 1\Big),
\end{aligned}$$

where we applied conjugation by the operator in $\mathrm{End}(\widetilde{\mathcal{V}}^{\otimes k})$ which reverse the order of tensor factors. Now the statement follows from (3.6) and (3.8) for $\mathrm{Y}(\mathfrak{gl}_{n|m})$. □

Lemma 4.8. *The isomorphism ϖ sends $I_{\boldsymbol{\xi},\boldsymbol{t},Q}$ for $\mathrm{Y}(\mathfrak{gl}_{m|n})$ to $\tilde{I}_{\bar{\boldsymbol{\xi}},\bar{\boldsymbol{t}},\mathrm{Q}}$ for $\mathrm{Y}(\mathfrak{gl}_{n|m})$.*

Proof. It suffices to check that ϖ maps $\mathcal{J}^{a,i}_{\boldsymbol{\xi},Q}(\boldsymbol{t})$ in (3.12) for $\mathrm{Y}(\mathfrak{gl}_{m|n})$ to $-\tilde{\mathcal{J}}^{a',j}_{\bar{\boldsymbol{\xi}},\mathrm{Q}}(\bar{\boldsymbol{t}})$ for $\mathrm{Y}(\mathfrak{gl}_{n|m})$, where $j=\xi^a+1-i$. Observing that the sequence of polynomials associated to $\bar{\boldsymbol{t}}$ and $\bar{\boldsymbol{\xi}}$ is $\bar{\boldsymbol{y}}=(y_{N-1},\dots,y_1)$ and κ_a for the former case corresponds to $-\tilde{\kappa}_{a'}$ for the later one, then the lemma is straightforward. □

4.3. **Proof of Theorem 3.3.** We prove Theorem 3.3 by induction on m. We first establish the base case $m=0$. By (3.5), transfer matrices $\mathfrak{T}_{k,Q}(u)$ associated to symmetrizers can be expressed in terms of transfer matrices $\mathcal{T}_{l,Q}(u)$ associated to antisymmetrizers. Now Theorem 3.3 for the base case ($m=0$) follows from the observations from Section 4.2 by applying the isomorphism ϖ to [MTV06, Theorem 5.4].

For the rest of the induction process, the procedure is almost parallel to that of [MTV06, Section 11], cf. [BR08]. We provide the details for completeness.

Since $\mathcal{V}=\mathbb{C}\mathbf{v}_1\oplus\mathcal{W}$ and $\mathbf{v}_1$ is even, one obtains $\mathcal{V}^{\wedge k}=(\mathbf{v}_1\wedge\mathcal{W}^{\wedge(k-1)})\oplus\mathcal{W}^{\wedge k}$, where the first summand is spanned by vectors of the form $\mathbf{v}_1\wedge\mathbf{v}_{a_1}\wedge\cdots\wedge\mathbf{v}_{a_{k-1}}$ with $2\leqslant a_1<\cdots<a_b\leqslant m<a_{b+1}\leqslant\cdots\leqslant a_{k-1}\leqslant N$, while the second one is spanned by vectors of the form $\mathbf{v}_{a_1}\wedge\cdots\wedge\mathbf{v}_{a_k}$ with $1\leqslant a_1<\cdots<a_b\leqslant m<a_{b+1}\leqslant\cdots\leqslant a_k\leqslant N$, both for various b. We also identify $\mathcal{W}^{\wedge(k-1)}$ with $(\mathbf{v}_1\wedge\mathcal{W}^{\wedge(k-1)})$ by $\mathbf{x}\mapsto\mathbf{v}_1\wedge\mathbf{x}$.

The R-matrix $R_{\wedge k,\wedge 1}(u)$, see Section 2.4, as an operator on $\mathcal{V}^{\wedge k}\otimes\mathcal{V}$ preserves the subspaces

$$(\mathbf{v}_1\wedge\mathcal{W}^{\wedge(k-1)})\oplus\mathbb{C}\mathbf{v}_1,\quad \big((\mathbf{v}_1\wedge\mathcal{W}^{\wedge(k-1)})\otimes\mathcal{W}\big)\oplus(\mathcal{W}^{\wedge k}\otimes\mathbb{C}\mathbf{v}_1),\quad \mathcal{W}^{\wedge k}\otimes\mathcal{W}.$$

Moreover, we have

$$\begin{gathered}R_{\wedge k,\wedge 1}(u)\big|_{(\mathbf{v}_1\wedge\mathcal{W}^{\wedge(k-1)})\oplus\mathbb{C}\mathbf{v}_1}=u+1,\quad R_{\wedge k,\wedge 1}(u)\big|_{\mathcal{W}^{\wedge k}\otimes\mathcal{W}}=R^{\langle\mathcal{N}-1\rangle}_{\wedge k,\wedge 1}(u),\\ R_{\wedge k,\wedge 1}(u)\big|_{\big((\mathbf{v}_1\wedge\mathcal{W}^{\wedge(k-1)})\otimes\mathcal{W}\big)\oplus(\mathcal{W}^{\wedge k}\otimes\mathbb{C}\mathbf{v}_1)}=\begin{pmatrix}R^{\langle\mathcal{N}-1\rangle}_{\wedge(k-1),\wedge 1}(u) & \tilde{\mathsf{S}}^t\\ \tilde{\mathsf{S}} & u\end{pmatrix},\end{gathered}\tag{4.5}$$

where $\tilde{\mathsf{S}}((\mathbf{v}_1\wedge\mathbf{x})\otimes\mathbf{w})=(-1)^{|\mathbf{x}||\mathbf{w}|}(\mathbf{w}\wedge\mathbf{x})\otimes\mathbf{v}_1$.

Regard $T(u)$ and $T^{\wedge k}(u)$, see (2.16), as matrices over the super Yangian $\mathrm{Y}(\mathfrak{gl}_{m|n})$ and consider the block decomposition induced by the decompositions $\mathcal{V}=\mathbb{C}\mathbf{v}_1\oplus\mathcal{W}$ and $\mathcal{V}^{\wedge k}=(\mathbf{v}_1\wedge\mathcal{W}^{\wedge(k-1)})\oplus\mathcal{W}^{\wedge k}$,

$$T(u)=\begin{pmatrix}A(u) & B(u)\\ C(u) & D(u)\end{pmatrix},\quad T^{\wedge k}(u)=\begin{pmatrix}\widehat{A}(u) & \widehat{B}(u)\\ \widehat{C}(u) & \widehat{D}(u)\end{pmatrix}.\tag{4.6}$$

For example, $A(u)=T_{11}(u)$, $B(u)=\sum_{a=2}^N E_{1a}\otimes T_{1a}(u)$, $D(u)=\sum_{a,b=2}^N E_{ab}\otimes T_{ab}(u)$.

We use the following convenient notations. Denote by $\mathrm{HY}(L,M)$ the superspace $\mathrm{Hom}(L,M)\otimes\mathrm{Y}(\mathfrak{gl}_{m|n})$ of matrices with noncommuting entries. Call L the domain of those matrices. For instance, the coefficients of the series B, D, $\widehat{A}$, $\widehat{B}$, $\widehat{D}$ belong to $\mathrm{HY}(\mathcal{W},\mathbb{C})$, $\mathrm{HY}(\mathcal{W},\mathcal{W})$, $\mathrm{HY}(\mathcal{W}^{\wedge(k-1)},\mathcal{W}^{\wedge(k-1)})$, $\mathrm{HY}(\mathcal{W}^{\wedge k},\mathcal{W}^{\wedge(k-1)})$, $\mathrm{HY}(\mathcal{W}^{\wedge k},\mathcal{W}^{\wedge k})$. In particular, if $k=1$, then $\widehat{X}(u)=X(u)$ for $X=A,B,C,D$.

Renormalize R-matrices,

$$\overline{R}(u)=\frac{1}{u}R^{\langle\mathcal{N}-1\rangle}(u),\quad \widetilde{R}(u)=\frac{1}{u+1}R^{\langle\mathcal{N}-1\rangle}_{\wedge k,\wedge 1}(u),\quad \widehat{R}(u)=\frac{1}{u}R^{\langle\mathcal{N}-1\rangle}_{\wedge k,\wedge 1}(u)$$

and define an even linear map

$$\mathsf{S}:\mathcal{W}^{\wedge(k-1)}\otimes\mathcal{W}\to\mathcal{W}^{\wedge k},\quad \mathsf{S}(\mathbf{x}\otimes\mathbf{w})=(-1)^{|\mathbf{x}||\mathbf{w}|}\mathbf{w}\wedge\mathbf{x}.$$

We have equalities

$$\widehat{B}^{[1]}(u)\widehat{B}^{[2]}(v) = \frac{u-v}{u-v+1}\widehat{B}^{[2]}(v)\widehat{B}^{[1]}(v)\widehat{R}^{(12)}(u-v) \tag{4.7}$$

$$\widehat{A}^{(1)}(u)\widehat{B}^{[2]}(v) = \widehat{B}^{[2]}(v)\widehat{A}^{(1)}(u)\widetilde{R}^{(12)}(u-v-1) + \frac{1}{u-v}\widehat{B}(u)\mathsf{S}^{[12]}A(v), \tag{4.8}$$

$$\widehat{D}^{(1)}(u)\widehat{B}^{[2]}(v) = \widehat{B}^{[2]}(v)\widehat{D}^{(1)}(u)\widehat{R}^{(12)}(u-v) - \frac{1}{u-v}\mathsf{S}\widehat{B}^{[1]}(u)D^{(2)}(v), \tag{4.9}$$

in $\mathrm{HY}(\mathcal{W}^{\wedge k}\otimes\mathcal{W},\mathcal{W}^{\wedge(k-1)})$, $\mathrm{HY}(\mathcal{W}^{\wedge(k-1)}\otimes\mathcal{W},\mathcal{W}^{\wedge(k-1)})$, and $\mathrm{HY}(\mathcal{W}^{\wedge k}\otimes\mathcal{W},\mathcal{W}^{\wedge k})$, respectively, where the superscripts in brackets indicate which tensor factors are domains of the corresponding matrices. In particular, if $k=1$, we have

$$B^{[1]}(u)B^{[2]}(v) = \frac{u-v}{u-v+1}B^{[2]}(v)B^{[1]}(v)\overline{R}^{(12)}(u-v) \tag{4.10}$$

$$A^{(1)}(u)B(v) = \frac{u-v-1}{u-v}B(v)A^{(1)}(u) + \frac{1}{u-v}B(u)A(v), \tag{4.11}$$

$$D^{(1)}(u)B^{[2]}(v) = B^{[2]}(v)D^{(1)}(u)\overline{R}^{(12)}(u-v) - \frac{1}{u-v}B^{[1]}(u)D^{(2)}(v), \tag{4.12}$$

see also [BR08, Equations (4.6)-(4.8)].

Let $\check{R}(u) = (u+(-1)^{|2|})^{-1}P^{\langle\mathcal{N}-1\rangle}R^{\langle\mathcal{N}-1\rangle}(u)$. For a function $f(u_1,\dots,u_r)$ with values in matrices with the domain $\mathcal{W}^{\otimes r}$ and a simple permutation $(i,i+1)$, $1\leqslant i<r$, set

$${}^{(i,i+1)}f(u_1,\dots,u_r) = f(u_1,\dots,u_{i-1},u_{i+1},u_i,u_{i+2},\dots,u_r)\check{R}^{(i,i+1)}(u_i-u_{i+1}). \tag{4.13}$$

Note that the matrix $\check{R}(u)$ satisfies $\check{R}(u)\check{R}(-u)=1$ and

$$\check{R}^{(12)}(u-v)\check{R}^{(23)}(u)\check{R}^{(12)}(v) = \check{R}^{(23)}(v)\check{R}^{(12)}(u)\check{R}^{(23)}(u-v).$$

Due to this, (4.13) extends to an action of the symmetric group $\mathfrak{S}_r$ on functions $f(u_1,\dots,u_r)$ with values in matrices with the domain $\mathcal{W}^{\otimes r}$, $f\mapsto {}^{\sigma}f$, $\sigma\in\mathfrak{S}_r$. Thanks to (4.10), the expression $B^{[1]}(u_1)\cdots B^{[r]}(u_r)$ is invariant under this action of $\mathfrak{S}_r$.

In general, for a function $f(u_1,\dots,u_r)$ with values in matrices with the domain $\mathcal{W}^{\otimes r}$, define

$${}^{R}\mathrm{Sym}_r f(u_1,\dots,u_r) = \sum_{\sigma\in\mathfrak{S}_r}{}^{\sigma}f(u_1,\dots,u_r).$$

Proposition 4.9 ([MTV06, Proposition 11.5])**.** *We have*

$$\begin{aligned}&\widehat{A}^{(0)}(u)B^{[1]}(u_1)\cdots B^{[r]}(u_r)\\ &= B^{[1]}(u_1)\cdots B^{[r]}(u_r)\widehat{A}^{(0)}\widetilde{R}^{(0r)}(u-u_r-1)\cdots\widetilde{R}^{(01)}(u-u_1-1)\\ &\quad+\frac{1}{(r-1)!}\widehat{B}(u){}^{R}\mathrm{Sym}_r\Big(\frac{1}{u-u_1}\prod_{i=2}^{r}\frac{u_1-u_i-1}{u_1-u_i}\mathsf{S}^{[01]}B^{[2]}(u_2)\cdots B^{[r]}(u_r)A(u_1)\Big),\end{aligned} \tag{4.14}$$

$$\begin{aligned}&\widehat{D}^{(0)}(u)B^{[1]}(u_1)\cdots B^{[r]}(u_r)\\ &= B^{[1]}(u_1)\cdots B^{[r]}(u_r)\widehat{D}^{(0)}\widehat{R}^{(0r)}(u-u_r)\cdots\widehat{R}^{(01)}(u-u_1)\\ &\quad-\frac{1}{(r-1)!}\mathsf{S}\widehat{B}^{[0]}(u){}^{R}\mathrm{Sym}_r\Big(\frac{1}{u-u_1}B^{[2]}(u_2)\cdots B^{[r]}(u_r)\\ &\qquad\qquad\times D^{(1)}(u_1)\overline{R}^{(1r)}(u_1-u_r)\cdots\overline{R}^{(12)}(u_1-u_2)\Big),\end{aligned} \tag{4.15}$$

where the tensor products are counted by $0, 1, \ldots, r$. $\square$

We also need the following statements. Note that formulas here are slightly different from those in [MTV06] as we are using the opposite coproduct, see (4.2).

Lemma 4.10 ([MTV06, Lemma 11.6])**.** *We have*

$$D^{(0)}(u)\overline{R}^{(01)}(u-u_r)\cdots\overline{R}^{(0r)}(u-u_1)=\psi(u_1,\ldots,u_r)\big(T^{\langle\mathcal{N}-1\rangle}(u)\big),$$
$$\widehat{D}^{(0)}(u)\widehat{R}^{(01)}(u-u_r)\cdots\widehat{R}^{(0r)}(u-u_1)=\psi(u_1,\ldots,u_r)\Big(\big(T^{\langle\mathcal{N}-1\rangle}(u)\big)^{\wedge k}\Big). \qquad \square$$

Lemma 4.11 ([MTV06, Lemma 11.7])**.** *For any* $X\in \mathrm{Y}(\mathfrak{gl}_{\mathcal{N}-1})$*, we have*

$$A(u)\psi(u_1,\ldots,u_r)(X)\simeq\psi(u_1,\ldots,u_r)(X)A(u), \tag{4.16}$$
$$\widehat{A}^{(0)}(u)\widetilde{R}^{(01)}(u-u_r-1)\cdots\widetilde{R}^{(0r)}(u-u_1-1)\psi(u_1,\ldots,u_r)(X) \tag{4.17}$$
$$\simeq\prod_{i=1}^{r}\frac{u-u_i-1}{u-u_i}A(u)\psi(u_1,\ldots,u_r)\Big(\big(T^{\langle\mathcal{N}-1\rangle}(u-1)\big)^{\wedge k}X\Big).$$

Proof. Let $\mathrm{Y}_\times(\mathfrak{gl}_{\mathcal{N}})$ be the left ideal of $\mathrm{Y}(\mathfrak{gl}_{\mathcal{N}})$ generated by the coefficients of the series $T_{a1}(u)$ for $2\leqslant a\leqslant N$. Note that $\mathrm{Y}_\times(\mathfrak{gl}_{\mathcal{N}})$ is a subideal of $\mathrm{Y}_+(\mathfrak{gl}_{\mathcal{N}})$. It is clear from the definition relations (2.6) that for any $Z\in\mathrm{Y}(\mathfrak{gl}_{\mathcal{N}-1})$ and $C\in\mathrm{Y}_\times(\mathfrak{gl}_{\mathcal{N}})$, the coefficients of $[T_{11}^{\langle\mathcal{N}\rangle}(u),\psi(Z)]$ and $C\psi(Z)$ belong to $\mathrm{Y}_\times(\mathfrak{gl}_{\mathcal{N}})$. Therefore (4.16) follows from the fact that $A(u)=T_{11}^{\langle\mathcal{N}\rangle}(u)$ and $\psi(u_1,\ldots,u_r)(X)\in\psi(Y(\mathfrak{gl}_{\mathcal{N}-1}))$.

It follows from [MR14, Proposition 2 & Remark 2.4] that the coefficients of entries of the matrix $\widehat{A}(u)-A(u)D^{\wedge k-1}(u-1)$ are in $\mathrm{Y}_\times(\mathfrak{gl}_{\mathcal{N}})$. Therefore, by Lemma 4.10, we have

$$\begin{aligned}&\widehat{A}^{(0)}(u)\widetilde{R}^{(01)}(u-u_r-1)\cdots\widetilde{R}^{(0r)}(u-u_1-1)\psi(u_1,\ldots,u_r)(X)\\ \simeq\ & A(u)\big(D^{\wedge k-1}(u-1)\big)^{[0]}\widehat{R}^{(01)}(u-u_r-1)\cdots\widehat{R}^{(0r)}(u-u_1-1)\psi(u_1,\ldots,u_r)(X)\\ \simeq\ & A(u)\psi(u_1,\ldots,u_r)\Big(\big(T^{\langle\mathcal{N}-1\rangle}(u-1)\big)^{\wedge k-1}\Big)\psi(u_1,\ldots,u_r)(X)\prod_{i=1}^{r}\frac{u-u_i-1}{u-u_i}\\ \simeq\ & A(u)\psi(u_1,\ldots,u_r)\Big(\big(T^{\langle\mathcal{N}-1\rangle}(u-1)\big)^{\wedge k-1}X\Big)\prod_{i=1}^{r}\frac{u-u_i-1}{u-u_i}.\end{aligned}$$

Here we also used the fact $\psi(u_1,\ldots,u_r)$ is a homomorphism of superalgebras. $\square$

Now we are ready to finish the proof of Theorem 3.5. Let $Q=\sum_{a=1}^{N}Q_aE_{aa}^{\langle\mathcal{N}\rangle}\in\mathrm{End}(\mathcal{V})$ and $\overline{Q}=\sum_{a=1}^{N-1}Q_{a+1}E_{a+1,a+1}^{\langle\mathcal{N}\rangle}\in\mathrm{End}(\mathcal{W})$. Set $\widetilde{Q}=\overline{Q}^{\wedge(k-1)}$ and $\widehat{Q}=\overline{Q}^{\wedge k}$. By the definition of transfer matrices, see (3.1), we have

$$\mathcal{T}_{k,Q}=Q_1\mathrm{str}_{\mathcal{W}^{\wedge(k-1)}}(\widetilde{Q}\widehat{A}(u))+\mathrm{str}_{\mathcal{W}^{\wedge k}}(\widehat{Q}\widehat{D}(u)). \tag{4.18}$$

Set $r=\xi^1$ and $u_i=t_i^1$, $1\leqslant i\leqslant r$. We have

$$\begin{aligned}&\mathcal{T}_{k,Q}(u)B^{[1]}(u_1)\cdots B^{[r]}(u_r) \qquad (4.19)\\ =&\prod_{1\leqslant i\leqslant r}^{\longrightarrow}B^{[i]}(u_i)\Big(Q_1(\mathrm{str}_{\mathcal{W}^{\wedge(k-1)}}\otimes\mathrm{id}^{\otimes r})\big(\widetilde{Q}^{(0)}\widehat{A}^{(0)}(u)\prod_{1\leqslant j\leqslant r}^{\longleftarrow}\widetilde{R}^{(0j)}(u-u_j-1)\big)\end{aligned}$$

$$+ (\mathrm{str}_{\mathcal{W}^{\wedge k}} \otimes \mathrm{id}^{\otimes r})\big(\widehat{Q}^{(0)} \widehat{D}^{(0)}(u) \prod_{1\leqslant j\leqslant r}^{\longleftarrow} \widehat{R}^{(0j)}(u-u_j)\big)\Big)$$

$$+ \frac{1}{(r-1)!}\,{}^{R}\mathrm{Sym}_{u_1,\dots,u_r}^{(1,\dots,r)}\Big[\frac{1}{u-u_1}\mathcal{B}_Q^{[1]}(u)\prod_{2\leqslant i\leqslant r}^{\longrightarrow} B^{[i]}(u_i)\Big(Q_1\prod_{j=2}^{r}\frac{u_1-u_j-1}{u-u_j}A(u_1)$$
$$- \overline{Q}^{(1)} D^{(1)}(u_1)\prod_{2\leqslant j\leqslant r}^{\longleftarrow}\overline{R}^{(1j)}(u_1-u_j)\Big)\Big],$$

where $\mathcal{B}_Q(u) = (\mathrm{str}_{\mathcal{W}^{\wedge(k-1)}} \otimes \mathrm{id})\big(\overline{Q}^{\wedge(k-1)}\widehat{B}(u)\mathsf{S}\big)$ and the tensor factors for the products under the traces are counted by $0, 1, \dots, r$. Here we also used the equality

$$(\mathrm{str}_{\mathcal{W}^{\wedge k}} \otimes \mathrm{id})\big(\overline{Q}^{\wedge k}\mathsf{S}(\widehat{B}(u)\otimes \mathrm{id})\big) = \mathcal{B}_Q(u)\overline{Q}$$

which follows from the supercyclicity of the supertrace and the formula

$$\overline{Q}^{\wedge k}\mathsf{S} = \mathsf{S}(\overline{Q}^{\wedge(k-1)}\otimes \overline{Q}).$$

Note that $\mathcal{B}_Q(u) = B(u)$ if $k=1$.

For an expression $f(v)$ set $\operatorname{res}_{v=w} f(v) = \big((v-w)f(v)\big)\big|_{v=w}$ if the substitution makes sense. By Lemma 4.10 and the equality $\operatorname{res}_{v=u_1}\big(\overline{R}(v-u_1)\big) = P^{\langle \mathcal{N}-1\rangle}$, we have

$$\begin{aligned}
\overline{Q}^{(1)} D^{(1)}(u_1)\prod_{2\leqslant j\leqslant r}^{\longleftarrow}\overline{R}^{(1j)}(u_1-u_j) &= \operatorname*{res}_{v=u_1}\Big((\mathrm{str}_{\mathcal{W}}\otimes\mathrm{id})(\overline{Q}^{(0)}D^{(0)}(v)\prod_{1\leqslant j\leqslant r}^{\longleftarrow}\overline{R}^{(0j)}(v-u_j))\Big)\\
&= \operatorname*{res}_{v=u_1}\Big((\mathrm{str}_{\mathcal{W}}\otimes\mathrm{id})(\overline{Q}^{(0)}D^{(0)}(v)\prod_{1\leqslant j\leqslant r}^{\longleftarrow}\overline{R}^{(0,r+1-j)}(v-u_j))\Big)\\
&= \operatorname*{res}_{v=u_1}\Big(\psi(u_1,\dots,u_r)\big(\mathcal{T}_{1,\overline{Q}}^{\langle\mathcal{N}-1\rangle}(v)\big)\Big).
\end{aligned}$$

In the second equality, we used the super cyclicity of super trace which allows us to permute factors by conjugating the super flip operators $P^{(i,j)}$. The same will also be used in the sequel which we shall not write explicitly.

Therefore, for any $X \in \mathrm{Y}(\mathfrak{gl}_{\mathcal{N}-1})$, by Lemma 4.11 and (4.19), we have

$$\begin{aligned}
&\mathcal{T}_{k,Q}(u)B^{[1]}(u_1)\cdots B^{[r]}(u_r)\psi(u_1,\dots,u_r)(X) \qquad (4.20)\\
&\simeq \prod_{1\leqslant i\leqslant r}^{\longrightarrow} B^{[i]}(u_i)\Big(\psi(u_1,\dots,u_r)\big(\mathcal{T}_{k-1,\overline{Q}}^{\langle\mathcal{N}-1\rangle}(u-1)X\big)Q_1A(u)\prod_{j=1}^{r}\frac{u-u_j-1}{u-u_j}\\
&\qquad + \psi(u_1,\dots,u_r)\big(\mathcal{T}_{k,\overline{Q}}^{\langle\mathcal{N}-1\rangle}(u)X\big)\Big)\\
&+ \frac{1}{(r-1)!}\,{}^{R}\mathrm{Sym}_{u_1,\dots,u_r}^{(1,\dots,r)}\Big[\frac{1}{u-u_1}\mathcal{B}_Q^{[1]}(u)\prod_{2\leqslant i\leqslant r}^{\longrightarrow} B^{[i]}(u_i)\\
&\times\Big(\psi(u_1,\dots,u_r)(X)Q_1A(u_1)\prod_{j=2}^{r}\frac{u_1-u_j-1}{u-u_j} - \operatorname*{res}_{v=u_1}\big(\psi(u_1,\dots,u_r)\big(\mathcal{T}_{1,\overline{Q}}^{\langle\mathcal{N}-1\rangle}(v)X\big)\big)\Big)\Big].
\end{aligned}$$

Then we apply both sides of (4.20) to the vector $\mathbf{w}_1^{\otimes r}$, set $X = \mathbb{B}_{\boldsymbol{\xi}}^{\langle\mathcal{N}-1\rangle}(\bar{\boldsymbol{t}})$, see Proposition 4.3, and employ the induction assumption. The first step amounts to replacing $\psi(u_1,\dots,u_r)$ by $\widetilde{\psi}(u_1,\dots,u_r)$, see Section 4.1, and changing ${}^{R}\mathrm{Sym}^{1,\dots,r}_{u_1,\dots,u_r}$ to the ordinary symmetrization $\mathrm{Sym}_{u_1,\dots,u_r}$ due to the fact that $\check{R}(u)\mathbf{w}_1\otimes\mathbf{w}_1=\mathbf{w}_1\otimes\mathbf{w}_1$.

We discuss the next two steps. Recall that $A(u)=T_{11}(u)$, $r=\xi^1$, and $u_i=t^1_i$, $1\leqslant i\leqslant r$. By Lemma 2.1, 4.1, and

$$\pi(x)(T^{\langle\mathcal{N}-1\rangle}_{aa}(u))=\Big(1+\frac{\kappa_2\delta_{1a}}{u-x}\Big)\mathbf{w}_1,\quad 1\leqslant a\leqslant N-1,$$

we have

$$\widetilde{\psi}(t^1_1,\dots,t^1_{\xi^1})(T^{\langle\mathcal{N}-1\rangle}_{aa}(u))=T_{a+1,a+1}(u)\otimes\mathbf{w}_1^{\otimes r}\prod_{i=1}^{\xi^1}\Big(1+\frac{\kappa_2\delta_{1a}}{u-t^1_i}\Big),\quad 1\leqslant a\leqslant N-1. \tag{4.21}$$

Then (4.20) becomes

$$\begin{aligned}&\mathcal{T}_{k,Q}(u)\mathbb{B}_{\boldsymbol{\xi}}(\boldsymbol{t}) \qquad (4.22)\\ &\simeq\mathbb{B}_{\boldsymbol{\xi}}(\boldsymbol{t})\Big(Q_1T_{11}(u)\prod_{j=1}^{\xi^1}\frac{u-u_j-1}{u-u_j}\sum_{\boldsymbol{a}}\prod_{j=2}^{k}\kappa_{a_j}\mathcal{X}^{a_j}_{\boldsymbol{\xi},Q}(u-j+1;\boldsymbol{t})\\ &\qquad+\sum_{\boldsymbol{b}}\prod_{j=1}^{k}\kappa_{b_j}\mathcal{X}^{b_j}_{\boldsymbol{\xi},Q}(u-j+1;\boldsymbol{t})\Big)+\mathcal{U}_{\boldsymbol{\xi},k,Q}(u;\boldsymbol{t})\end{aligned}$$

where the sums are taken over all possible $\boldsymbol{a}=(2\leqslant a_2<\cdots<a_i<m+1\leqslant a_{i+1}\leqslant\cdots\leqslant a_k\leqslant N)$ and $\boldsymbol{b}=(2\leqslant b_1<\cdots<b_j<m+1\leqslant b_{j+1}\leqslant\cdots\leqslant b_k\leqslant N)$ for various $1\leqslant i\leqslant k$ and $0\leqslant j\leqslant k$, respectively, and

$$\begin{aligned}&\mathcal{U}_{\boldsymbol{\xi},k,Q}(u;\boldsymbol{t}) \qquad (4.23)\\ &\simeq\overrightarrow{\prod_{1\leqslant i\leqslant t^1_{\xi_1}}}B^{[i]}(t^1_i)\Big(\widetilde{\psi}(t^1_1,\dots,t^1_{\xi_1})\big(\mathcal{U}^{\langle\mathcal{N}-1\rangle}_{\bar{\boldsymbol{\xi}},k-1,\overline{Q}}(u-1;\bar{\boldsymbol{t}})\big)Q_1T_{11}(u)\prod_{j=1}^{\xi^1}\frac{u-t^1_j-1}{u-t^1_j}\\ &\qquad+\widetilde{\psi}(t^1_1,\dots,t^1_{\xi^1})\big(\mathcal{U}^{\langle\mathcal{N}-1\rangle}_{\bar{\boldsymbol{\xi}},k,\overline{Q}}(u;\bar{\boldsymbol{t}})\big)\Big)\\ &+\frac{1}{(\xi^1-1)!}\mathrm{Sym}^{(1,\cdots,\xi^1)}_{t^1_1,\dots,t^1_{\xi^1}}\Bigg[\frac{1}{u-t^1_1}\mathcal{B}^{[1]}_Q(u)\overrightarrow{\prod_{2\leqslant i\leqslant\xi^1}}B^{[i]}(t^1_i)\\ &\times\Bigg(\widetilde{\psi}(t^1_1,\dots,t^1_{\xi^1})(\mathbb{B}^{\langle\mathcal{N}-1\rangle}_{\bar{\boldsymbol{\xi}}}(\boldsymbol{t}))\\ &\times\Big(Q_1T_{11}(t^1_1)\prod_{j=2}^{\xi^1}\frac{t^1_1-t^1_j-1}{t^1_1-t^1_j}-\kappa_2Q_2T_{22}(u)\frac{y_1(t^1_1+\kappa_2)y_2(t^1_1-\kappa_2)}{y_1'(t^1_1)y_2(t^1_1)}\Big)\\ &\qquad-\operatorname*{res}_{v=u_1}\Big(\widetilde{\psi}(t^1_1,\dots,t^1_{\xi^1})\big(\mathcal{U}^{\langle\mathcal{N}-1\rangle}_{\bar{\boldsymbol{\xi}},1,\overline{Q}}(v;\bar{\boldsymbol{t}})\big)\Big)\Bigg)\Bigg].\end{aligned}$$

Here κ_2 in $y_1(t^1_1+\kappa_2)$ comes from (4.21). Finally, using the induction hypothesis that the expressions $\mathcal{U}^{\langle\mathcal{N}-1\rangle}_{\bar{\boldsymbol{\xi}},j,\overline{Q}}(v;\bar{\boldsymbol{t}})$ are contained in $I^{\langle\mathcal{N}-1\rangle}_{\bar{\boldsymbol{\xi}},j,\overline{Q}}(v;\bar{\boldsymbol{t}})$, Lemmas 2.1, 2.3, 4.1, 4.2, and formulas (3.13), (4.1), (4.2), we conclude that there exists an element $\mathcal{U}_{\boldsymbol{\xi},k,Q}(u;\boldsymbol{t})$ in $I_{\boldsymbol{\xi},\boldsymbol{t},k,Q}$ satisfying (4.23), completing the proof of Theorem 3.3 from (4.22).

Proof of Proposition 3.7. Let $C(u) = \sum_{a=2}^{N} E_{a1} \otimes T_{a1}(u)$ be the left bottom block of $T(u)$ in (4.6). Let $C(u) = \sum_{s=1}^{\infty} C_s u^{-s}$. Similar to the proof of [MTV06, Proposition 6.2], we have

$$C_1 B(u) - B(u) C_1 = \kappa_1 (A(u) - D(u)) = A(u) - D(u).$$

The rest is parallel to that of [MTV06, Proposition 6.2] which we shall skip the detail. □

5. Gaudin models

By taking the classical limits, we obtain the corresponding result for Gaudin models in this section. We start with preparing notations for the Gaudin case.

5.1. **Current superalgebra.** Let $\mathfrak{gl}_{m|n}[x]$ be the Lie superalgebra $\mathfrak{gl}_{m|n} \otimes \mathbb{C}[x]$ of $\mathfrak{gl}_{m|n}$-valued polynomials in x with the point-wise supercommutator. Call $\mathfrak{gl}_{m|n}[x]$ the *current superalgebra* of $\mathfrak{gl}_{m|n}$.

We write $e_{ab}^{\{r\}}$ for $e_{ab} \otimes x^r$, $r \in \mathbb{Z}_{\geqslant 0}$. A basis of $\mathfrak{gl}_{m|n}[x]$ is given by $e_{ab}^{\{r\}}$, $1 \leqslant a, b \leqslant N$ and $r \in \mathbb{Z}_{\geqslant 0}$. They satisfy the supercommutator relations

$$[e_{ab}^{\{r\}}, e_{cd}^{\{s\}}] = \delta_{bc} e_{ad}^{\{r+s\}} - (-1)^{(|a|+|b|)(|c|+|d|)} \delta_{ad} e_{cb}^{\{r+s\}}.$$

We identify $\mathfrak{gl}_{m|n}$ with the subalgebra $\mathfrak{gl}_{m|n} \otimes 1$ of constant polynomials in $\mathfrak{gl}_{m|n}[x]$, that is we identify e_{ab} in $\mathfrak{gl}_{m|n}$ with $e_{ab}^{\{0\}}$ in $\mathfrak{gl}_{m|n}[x]$. Denote by $\mathrm{U}(\mathfrak{gl}_{m|n}[x])$ the universal enveloping superalgebra of $\mathfrak{gl}_{m|n}[x]$. Let $\mathfrak{n}_{\pm}[x]$ be the corresponding subalgebras in $\mathfrak{gl}_{m|n}[x]$, see (2.1).

We say that a vector in a $\mathfrak{gl}_{m|n}[x]$-module is called a *weight singular vector* if $\mathfrak{n}_{+}[x] v = 0$ and v is an eigenvector for all $e_{aa}^{\{s\}}$, $1 \leqslant a \leqslant N$, $s \in \mathbb{Z}_{>0}$.

For any $\mathfrak{gl}_{m|n}$-module M and $z \in \mathbb{C}$, we have the evaluation $\mathfrak{gl}_{m|n}[x]$-module $M(\!|z|\!)$ at the evaluation point z with the action given by $e_{ab}^{\{s\}}\big|_{M(\!|z|\!)} = z^s e_{ab}\big|_{M}$.

Consider the generating series and generating matrix

$$L_{ab}(u) = (-1)^{|b|} \sum_{s=0}^{\infty} e_{ba}^{\{s\}} u^{-s-1} \in \mathrm{U}(\mathfrak{gl}_{m|n}[x])[[u^{-1}]], \quad 1 \leqslant a, b \leqslant N,$$

$$L(u) = \sum_{a,b=1}^{N} E_{ab} \otimes L_{ab}(u) \in \mathrm{End}(\mathcal{V}) \otimes \mathrm{U}(\mathfrak{gl}_{m|n}[x])[[u^{-1}]].$$

Note that the indices in the generating series are flipped and the signs in the generating matrix are added so that it matches the evaluation map of super Yangian we used in (2.10).

5.2. **Gaudin transfer matrices.** To define Gaudin transfer matrices, we first recall basics about pseudo-differential operators.

Let $\mathcal{A}$ be a differential superalgebra with an even derivation $\partial : \mathcal{A} \to \mathcal{A}$. For $r \in \mathbb{Z}_{>0}$, denote the r-th derivative of $a \in \mathcal{A}$ by $a_{[r]}$. Define the *superalgebra of pseudo-differential operators* $\mathcal{A}((\partial^{-1}))$ as follows. Elements of $\mathcal{A}((\partial^{-1}))$ are Laurent series in ∂^{-1} with coefficients in $\mathcal{A}$, and the product is given by

$$\partial \partial^{-1} = \partial^{-1} \partial = 1, \quad \partial^r a = \sum_{s=0}^{\infty} \binom{r}{s} a_{[s]} \partial^{r-s}, \quad r \in \mathbb{Z}, \quad a \in \mathcal{A},$$

where

$$\binom{r}{s} = \frac{r(r-1)\cdots(r-s+1)}{s!}.$$

Let

$$\mathcal{A}_u^{m|n} = \mathrm{U}(\mathfrak{gl}_{m|n}[x])((u^{-1})) = \Big\{ \sum_{r=-\infty}^{s} g_r u^r,\ g_r \in \mathrm{U}(\mathfrak{gl}_{m|n}[x]),\ s \in \mathbb{Z} \Big\}$$

Fix a matrix $K = (K_{ab})_{1\leqslant a,b\leqslant N} \in \mathrm{End}(\mathcal{V})$, then the operator in $\mathrm{End}(\mathcal{V}) \otimes \mathcal{A}_u^{m|n}((\partial_u^{-1}))$,

$$\mathfrak{Z}_K(u,\partial_u) := \partial_u - K - L^{\dagger}(u) = \sum_{a,b=1}^{N} E_{ab} \otimes \big(\delta_{ab}\partial_u - K_{ab} - L_{ab}(u)(-1)^{|a||b|+|a|+|b|}\big)$$

is a Manin matrix, see [MR14, Lemma 3.1] and [HM20, Lemma 4.2].

Consider the quantum Berezinian $\mathrm{Ber}(\mathfrak{Z}_K(u,\partial_u))$ and expand it as an element in $\mathcal{A}_u^{m|n}((\partial_u^{-1}))$,

$$\mathfrak{D}_K(u,\partial_u) = \mathrm{Ber}(\partial_u - K - L^{\dagger}(u)) = \sum_{r=0}^{\infty} \mathcal{G}_{r,K}(u)\partial_u^{m-n-r}. \tag{5.1}$$

We call the series $\mathcal{G}_{r,K}(u) \in \mathcal{A}_u^{m|n}$, $r \in \mathbb{Z}_{\geqslant 0}$, the *Gaudin transfer matrices*.

Note that this family of series are different from that in [MR14], see Section 6.1. However, the coefficients of those two family of series generate the same subalgebra of $\mathrm{U}(\mathfrak{gl}_{m|n}[x])$ which we call the *Bethe subalgebra* of $\mathrm{U}(\mathfrak{gl}_{m|n}[x])$, see [HM20, Proposition 4.4].

The following properties about Gaudin transfer matrices are known.

Lemma 5.1 ([MR14]). *We have*

(1) $[\mathcal{G}_{r,K}(u), \mathcal{G}_{s,K}(v)] = 0$;
(2) *if K is the zero matrix, then the coefficients of $\mathcal{G}_{r,K}(u)$ commutes with the subalgebra $\mathrm{U}(\mathfrak{gl}_{m|n})$ of $\mathrm{U}(\mathfrak{gl}_{m|n}[x])$.*

5.3. **Bethe vectors.** Let $M_1,\dots,M_\ell$ be $\mathfrak{gl}_{m|n}$-modules, $\boldsymbol{z} = (z_1,\dots,z_\ell)$ a sequence of complex numbers. Consider the tensor product of evaluation $\mathfrak{gl}_{m|n}[x]$-modules $M(\!(\boldsymbol{z})\!) = M_1(\!(z_1)\!) \otimes \cdots \otimes M_\ell(\!(z_\ell)\!)$. Then we have

$$L_{ab}(u)\big|_{M(\boldsymbol{z})} = \kappa_b \sum_{i=1}^{\ell} \frac{e_{ba}^{(i)}}{u - z_i}$$

and the operator $\mathcal{G}^M_{r,K}(u;\boldsymbol{z}) = \mathcal{G}_{r,K}(u)|_{M(\boldsymbol{z})}$, for each $r \in \mathbb{Z}_{\geqslant 0}$, is a rational function in $u, \boldsymbol{z}$ with the denominator $\prod_{i=1}^{\ell}(u-z_i)^r$. We call the operators $\mathcal{G}^M_{r,K}(u;\boldsymbol{z})$ the *transfer matrices of the Gaudin model* on $M(\!(\boldsymbol{z})\!)$ associated with $\mathfrak{gl}_{m|n}$.

We are interested in the case when $M_1,\dots,M_\ell$ are highest weight $\mathfrak{gl}_{m|n}$-modules with highest weights $\Lambda_1,\dots,\Lambda_\ell$, where $\Lambda_i = (\Lambda_i^1,\dots,\Lambda_i^N)$, and highest weight vectors $v_1,\dots,v_\ell$. Set $\boldsymbol{\Lambda} = (\Lambda_1,\dots,\Lambda_\ell)$. In this case, the vector $v^+ = v_1\otimes\cdots\otimes v_\ell$ is a singular weight vector of $\mathrm{U}(\mathfrak{gl}_{m|n}[x])$ in $M(\!(\boldsymbol{z})\!)$,

$$L_{aa}(u)v^+ = v^+\kappa_a\Big(\sum_{i=1}^{\ell}\frac{\Lambda_i^a}{u-z_i}\Big), \quad 1\leqslant a\leqslant N. \tag{5.2}$$

Now we assume that $K = \sum_{a=1}^{N} K_a E_{aa} \in \mathrm{End}(\mathcal{V})$ is diagonal and use similar notations as in Section 3.2.

Let $\boldsymbol{\xi} = (\xi^1, \dots, \xi^{N-1})$ be a sequence of nonnegative integers. Consider an expression $\mathbb{F}_{\boldsymbol{\xi}}(\boldsymbol{t})$ in $|\boldsymbol{\xi}|$ variables $\boldsymbol{t} = (t_1^1, \dots, t_{\xi^1}^1, \dots, t_1^{N-1}, \dots, t_{\xi^{N-1}}^{N-1})$ with coefficients in $\mathrm{U}(\mathfrak{gl}_{m|n}[x])$ which will be defined later in (6.10) of Section 6.3. Here we only need to notice that $\mathbb{F}_{\boldsymbol{\xi}}(\boldsymbol{t})$ is obtained from $\mathbb{B}_{\boldsymbol{\xi}}(\boldsymbol{t})$ by taking certain gradation, see Proposition 6.9. Apply $\mathbb{F}_{\boldsymbol{\xi}}(\boldsymbol{t})$ to v^+ and renormalize it so that the function

$$\mathbb{F}_{\boldsymbol{\xi}}^{v^+}(\boldsymbol{t}; \boldsymbol{z}) = \mathbb{F}_{\boldsymbol{\xi}}(\boldsymbol{t}) v^+ \prod_{a=1}^{N-1} \prod_{i=1}^{\xi^a} \prod_{j=1}^{\ell} (t_i^a - z_\ell) \prod_{a=1}^{N-2} \prod_{i=1}^{\xi^a} \prod_{j=1}^{\xi^{a+1}} (t_j^{a+1} - t_i^a) \tag{5.3}$$

is a polynomial in $\boldsymbol{t}$, $\boldsymbol{z}$, see Section 6.3. We call $\mathbb{F}_{\boldsymbol{\xi}}^{v^+}(\boldsymbol{t}; \boldsymbol{z})$ an *off-shell Bethe vector* for the Gaudin model on $M(\!|\boldsymbol{z}|\!)$ associated with $\mathfrak{gl}_{m|n}$.

Let $\boldsymbol{y} = (y_1, \dots, y_{N-1})$ be the sequence of polynomials associated to $\boldsymbol{t}$ and $\boldsymbol{\xi}$. The system of algebraic equations in $|\boldsymbol{\xi}|$ variables $\boldsymbol{t}$,

$$\begin{aligned} K_a - K_{a+1} + \sum_{j=1}^{\ell} \frac{\kappa_a \Lambda_j^a - \kappa_{a+1} \Lambda_j^{a+1}}{t_i^a - z_j} + \frac{\kappa_a y_{a-1}'(t_i^a)}{y_{a-1}(t_i^a)} \\ - \frac{(\kappa_a + \kappa_{a+1}) y_a''(t_i^a)}{2 y_a'(t_i^a)} + \frac{\kappa_{a+1} y_{a+1}'(t_i^a)}{y_{a+1}(t_i^a)} = 0, \end{aligned} \tag{5.4}$$

$1 \leqslant a \leqslant N-1$, $1 \leqslant i \leqslant \xi^a$, is called the *Bethe ansatz equation*, see [MVY15, Equation (4.2)], which is usually written in the form,

$$\mathfrak{K}_{\boldsymbol{\xi},K}^{a,i}(\boldsymbol{t}; \boldsymbol{z}; \boldsymbol{\Lambda}) = 0, \tag{5.5}$$

where

$$\begin{aligned} \mathfrak{K}_{\boldsymbol{\xi},K}^{a,i}(\boldsymbol{t}; \boldsymbol{z}; \boldsymbol{\Lambda}) = K_a - K_{a+1} + \sum_{j=1}^{\ell} \frac{\kappa_a \Lambda_j^a - \kappa_{a+1} \Lambda_j^{a+1}}{t_i^a - z_j} + \sum_{j=1}^{\xi^{a-1}} \frac{\kappa_a}{t_i^a - t_j^{a-1}} \\ - \sum_{j=1, j\neq i}^{\xi^a} \frac{\kappa_a + \kappa_{a+1}}{t_i^a - t_j^a} + \sum_{j=1}^{\xi^{a+1}} \frac{\kappa_{a+1}}{t_i^a - t_j^{a+1}}, \end{aligned} \tag{5.6}$$

$1 \leqslant a < N$, and $1 \leqslant i \leqslant \xi^a$. We always assume that for a solution $\tilde{\boldsymbol{t}} = (\tilde{t}_1^1, \dots, \tilde{t}_{\xi^{N-1}}^{N-1})$ of system (5.5), any denominators in these equations does not vanish unless the corresponding numerator is zero.

When $\tilde{\boldsymbol{t}}$ is a solution of the Bethe ansatz equation (3.18), we say that the vector $\mathbb{F}_{\boldsymbol{\xi}}^{v^+}(\tilde{\boldsymbol{t}}; \boldsymbol{z})$ is an *on-shell Bethe vector*.

5.4. **Main results for Gaudin models.** For $1 \leqslant a \leqslant N$, define

$$\mathfrak{X}_{\boldsymbol{\xi},K}^a(u; \boldsymbol{t}; \boldsymbol{z}; \boldsymbol{\Lambda}) = K_a + \kappa_a \Big(\sum_{j=1}^{\ell} \frac{\Lambda_j^a}{u - z_j} + \frac{y_{a-1}'(u)}{y_{a-1}(u)} - \frac{y_a'(u)}{y_a(u)} \Big), \tag{5.7}$$

where $\boldsymbol{y} = (y_1, \dots, y_{N-1})$ is the sequence of polynomials associated to $\boldsymbol{t}$ and $\boldsymbol{\xi}$.

Theorem 5.2. *Let K be a diagonal matrix. If $M_1, \dots, M_\ell$ are highest weight $\mathfrak{gl}_{m|n}$-modules with highest weights $\Lambda_1, \dots, \Lambda_\ell$ and $\tilde{\boldsymbol{t}}$ is an isolated solution of the Bethe ansatz equation* (5.5)*, then we have*

$$\mathfrak{D}_K(u, \partial_u) \mathbb{F}_{\boldsymbol{\xi}}^{v^+}(\tilde{\boldsymbol{t}}; \boldsymbol{z}) = \mathbb{F}_{\boldsymbol{\xi}}^{v^+}(\tilde{\boldsymbol{t}}; \boldsymbol{z}) \overrightarrow{\prod_{1 \leqslant a \leqslant N}} \big(\partial_u - \mathfrak{X}_{\boldsymbol{\xi},K}^a(u; \tilde{\boldsymbol{t}}; \boldsymbol{z}; \boldsymbol{\Lambda})\big)^{\kappa_a},$$

where $\mathfrak{D}_K(u, \partial_u)$ is defined in (5.1)*.*

The theorem is proved in Section 6.4. It is an analog of [FFR94, Theorem 3] for supersymmetric case in type A. Note that the condition for the solution being isolated in Theorem 5.2 can be removed. If the solution is not isolated, then the statements can be proved similarly as in [MTV06, Theorems 8.6 & 9.2] as we have done for the XXX spin chain case, cf. Section 6.5. The theorem for the case of $m = n = 1$ was announced in [Lu22b, Theorem 4.11].

Note that the pseudo-differential operator

$$\mathfrak{D}_Q(u, \partial_u; \tilde{\boldsymbol{t}}; \boldsymbol{z}; \boldsymbol{\Lambda}) := \prod_{1 \leqslant a \leqslant N}^{\longrightarrow} \big(\partial_u - \mathfrak{X}^a_{\boldsymbol{\xi},K}(u; \tilde{\boldsymbol{t}}; \boldsymbol{z}; \boldsymbol{\Lambda})\big)^{\kappa_a} \tag{5.8}$$

was introduced in [HMVY19, Equation (6.5)].

Proposition 5.3 (cf. [MVY15, Theorem 4.3])**.** *Let K be the zero matrix. If $M_1, \dots, M_\ell$ are highest weight $\mathfrak{gl}_{m|n}$-modules with highest weights $\Lambda_1, \dots, \Lambda_\ell$ and $\tilde{\boldsymbol{t}}$ is an isolated solution of the Bethe ansatz equation (5.5), then the on-shell Bethe vector $\mathbb{F}^{v^+}_{\boldsymbol{\xi}}(\tilde{\boldsymbol{t}}; \boldsymbol{z})$ is a $\mathfrak{gl}_{m|n}$-singular vector in $M_1 \otimes \cdots \otimes M_\ell$ with weight*

$$\Big(\sum_{i=1}^{\ell} \Lambda_i^1 - \xi^1, \sum_{i=1}^{\ell} \Lambda_i^1 + \xi^1 - \xi^2, \dots, \sum_{i=1}^{\ell} \Lambda_i^N + \xi^{N-1}\Big).$$

The proposition is proved in Section 6.4.

6. More on Gaudin models

6.1. **More on transfer matrices.** Recall that $\mathcal{N} = m - n$ which is the supertrace of identity operator on $\mathcal{V}$ and also the super-dimension of $\mathcal{V}$. Note that here $\mathcal{N}$ may be negative. In the rest of this paper, our convention for ratios of factorials involving $\mathcal{N}$ is that we first assume $\mathcal{N}$ is a formal variable, then cancel common factors, and finally plug in $\mathcal{N} = m - n$.

Lemma 6.1. *For any $l \geqslant k$, and any distinct $i_1, \dots, i_k \in \{1, \dots, l\}$, we have*

$$\begin{aligned}(\mathrm{str}_{\mathcal{V}^{\otimes l}} \otimes \mathrm{id})(Q^{(i_1)} \cdots Q^{(i_k)} T^{(i_1, l+1)}(u) \cdots T^{(i_k, l+1)}(u - k + 1)\mathbb{A}^{(1\cdots l)}_{\{l\}})& \\ = \frac{k!(\mathcal{N}-k)!}{l!(\mathcal{N}-l)!}\mathcal{T}_{k,Q}(u).&\end{aligned}$$

Proof. If $l = k$ and $i_j = j$ for all $1 \leqslant j \leqslant k$, then the statement is equivalent to (3.1). For general cases, it follows from the equalities

$$P^{(ij)} Q^{(i)} T^{(i,l+1)} = Q^{(j)} T^{(j,l+1)} P^{(ij)}, \quad P^{(ij)} \mathbb{A}_{\{l\}} = \mathbb{A}_{\{l\}} P^{(ij)},$$

the cyclicity of supertrace, and the formula

$$(\mathrm{id}^{\otimes k} \otimes \mathrm{str}_{\mathcal{V}^{\otimes (l-k)}})\mathbb{A}_{\{l\}} = \frac{k!(\mathcal{N}-k)!}{l!(\mathcal{N}-l)!}\mathbb{A}_{\{k\}}.$$

∎

Remark 6.2. Note that since $k \leqslant l$, the denominator would not be zero after cancellation. However, unlike the even case, the number $\dfrac{k!(\mathcal{N}-k)!}{l!(\mathcal{N}-l)!}$ can be zero for certain l and $k \in \{1, \dots, l\}$.

Define another family of difference operators in $\mathrm{Y}(\mathfrak{gl}_{m|n})[[u^{-1},\partial_u]]$, cf. [Tal06, MTV06],

$$\mathcal{D}_{l,Q}(u,\partial_u)=(\mathrm{str}_{\mathcal{V}^{\otimes l}}\otimes\mathrm{id})\Big(\Big(\prod_{1\leqslant i\leqslant l}^{\longrightarrow}\big(1-Q^{(i)}T^{(i,l+1)}(u)e^{-\partial_u}\big)\Big)\mathbb{A}_{\{l\}}^{(1\cdots l)}\Big) \tag{6.1}$$

for $l\in\mathbb{Z}_{>0}$. By Lemma 6.1, we have the following corollary.

Corollary 6.3. *For $l\in\mathbb{Z}_{>0}$, we have*

$$\mathcal{D}_{l,Q}(u,\partial_u)=\frac{1}{(\mathcal{N}-l)!}\sum_{k=0}^{l}(-1)^k\frac{(\mathcal{N}-k)!}{(l-k)!}\mathcal{T}_{k,Q}(u)e^{-k\partial_u}.$$

Remark 6.4. It follows from (2.15) and $(\mathbb{A}_{\{l\}})^2=\mathbb{A}_{\{l\}}$ that

$$\prod_{1\leqslant i\leqslant l}^{\longrightarrow}\big(1-Q^{(i)}T^{(i,l+1)}(u)e^{-\partial_u}\big)\mathbb{A}_{\{l\}}^{(1\cdots l)}$$

$$=\mathbb{A}_{\{l\}}^{(1\cdots l)}\prod_{1\leqslant i\leqslant l}^{\longrightarrow}\big(1-Q^{(i)}T^{(i,l+1)}(u)e^{-\partial_u}\big)\mathbb{A}_{\{l\}}^{(1\cdots l)},$$

see also [MR14, Proposition 2.1]. □

There are also another family of Gaudin transfer matrices, see [MR14], defined as follows. For each $l\in\mathbb{Z}_{>0}$, consider the formal differential operator,

$$\mathfrak{D}_{l,K}(u,\partial_u)=(\mathrm{str}_{\mathcal{V}^{\otimes l}}\otimes\mathrm{id})\Big(\Big(\prod_{1\leqslant i\leqslant l}^{\longrightarrow}\big(\partial_u-K^{(i)}-L^{(i,l+1)}(u)\big)\Big)\mathbb{A}_{\{l\}}^{(1\cdots l)}\Big). \tag{6.2}$$

Let $\mathfrak{G}_{lk,K}(u)\in\mathrm{U}(\mathfrak{gl}_{m|n}[x])[[u^{-1}]]$, $l\in\mathbb{Z}_{>0}$ and $1\leqslant k\leqslant l$, be the coefficients of $\mathfrak{G}_{l,K}(u,\partial_u)$,

$$\mathfrak{D}_{l,K}(u,\partial_u)=\sum_{k=0}^{l}(-1)^k\mathfrak{G}_{lk,K}(u)\partial_u^{l-k}. \tag{6.3}$$

Let w be a formal variable. It is known from [MR14, Theorem 2.13] that

$$\mathrm{Ber}(1+w\mathfrak{Z}_K(u,\partial_u))=\sum_{k=0}^{\infty}w^k\mathfrak{D}_{l,K}(u,\partial_u). \tag{6.4}$$

6.2. **Filtration on $\mathrm{Y}(\mathfrak{gl}_{m|n})$.** Consider a filtered superalgebra A with an ascending filtration $\cdots\subset A_{s-1}\subset A_s\subset A_{s+1}\subset\cdots\subset A$. Denote by $\mathrm{gr}_s^A:A_s\to A_s/A_{s-1}$ the natural projection and identify the quotient spaces with the corresponding homogeneous subspaces in the graded superalgebra

$$\mathrm{gr}A=\bigoplus_{r\in\mathbb{Z}}A_r/A_{r-1}.$$

Then gr_s^A is regarded as a map from A_s to $\mathrm{gr}A$. We will simply write gr_s for gr_s^A when the superalgebra A is clear in the context. The superalgebra $\mathrm{End}(\mathcal{V})\otimes A$ also has a filtration induced from that on A.

The super Yangian $\mathrm{Y}(\mathfrak{gl}_{m|n})$ has a degree function defined by $\deg T_{ab}^{\{s\}}=s-1$ for $1\leqslant a,b\leqslant N$ and $s\in\mathbb{Z}_{>0}$. Then $\mathrm{Y}(\mathfrak{gl}_{m|n})$ is a filtered superalgebra with $\mathrm{Y}(\mathfrak{gl}_{m|n})_s$ being the subspace spanned by elements whose degrees are at most s. It is well known that $\mathrm{gr}(\mathrm{Y}(\mathfrak{gl}_{m|n}))=\mathrm{U}(\mathfrak{gl}_{m|n}[x])$ and $\mathrm{gr}_{s-1}(T_{ab}^{\{s\}})=(-1)^{|b|}e_{ba}^{\{s\}}$.

The filtration on $\mathrm{Y}(\mathfrak{gl}_{m|n})$ can be extended to the superalgebra $\mathrm{Y}(\mathfrak{gl}_{m|n})[[u^{-1},\partial_u]]$: $\deg u^{-1}=\deg\partial_u=-1$. Clearly, $\mathrm{gr}(\mathrm{Y}(\mathfrak{gl}_{m|n})[[u^{-1},\partial_u]])=\mathrm{U}(\mathfrak{gl}_{m|n}[x])[[u^{-1},\partial_u]]$. The series $T_{ab}(u)-\delta_{ab}\in\mathrm{Y}(\mathfrak{gl}_{m|n})[[u^{-1},\partial_u]]$ has degree -1 and

$$\mathrm{gr}_{-1}(T_{ab}(u)-\delta_{ab})=L_{ab}(u),\quad \mathrm{gr}_{-1}(T(u)-1)=L(u). \tag{6.5}$$

We assume further that Q in (3.1) is a series in $\mathrm{End}(\mathcal{V})[[\zeta]]$ instead of simply in $\mathrm{End}(\mathcal{V})$. Then transfer matrices are power series in u^{-1} and ζ with coefficients in $\mathrm{Y}(\mathfrak{gl}_{m|n})$. Hence we consider transfer matrices are elements in $\mathrm{Y}(\mathfrak{gl}_{m|n})[[u^{-1},\partial_u,\zeta]]$. Results and construction adapting to this new assumption naturally generalizes to the described setting. Extend further the filtration on $\mathrm{Y}(\mathfrak{gl}_{m|n})[[u^{-1},\partial_u]]$ to $\mathrm{Y}(\mathfrak{gl}_{m|n})[[u^{-1},\partial_u,\zeta]]$ by $\deg\zeta=-1$. Similarly, in the Gaudin model case, we assume K to be an element in $\mathrm{End}(\mathcal{V})[[\zeta]]$.

By convention, set $\mathcal{T}_{0,Q}(u)=1$. For any $k\in\mathbb{Z}_{\geqslant 0}$, set

$$\mathcal{S}_{k,Q}(u)=\frac{1}{(\mathcal{N}-k)!}\sum_{i=0}^{k}(-1)^{k-i}\frac{(\mathcal{N}-i)!}{(k-i)!}\mathcal{T}_{i,Q}(u).$$

Equivalently, for $k,l\in\mathbb{Z}_{\geqslant 0}$, $\mathcal{S}_{k,Q}(u)$ satisfy

$$\sum_{k=0}^{l}(-1)^{k}\frac{(\mathcal{N}-k)!}{(l-k)!}\mathcal{S}_{k,Q}(u)y^{l-k}=\sum_{i=0}^{l}(-1)^{i}\frac{(\mathcal{N}-i)!}{(l-i)!}\mathcal{T}_{i,Q}(u)(y+1)^{l-i}, \tag{6.6}$$

where y is a formal variale.

Proposition 6.5. *Let* $\deg(Q-1)\leqslant -1$ *with* $K=\mathrm{gr}_{-1}(Q-1)$. *Then* $\deg(\mathcal{S}_{k,Q}(u))=-k$ *for all* $k\in\mathbb{Z}_{\geqslant 0}$. *Moreover, for any* $l\in\mathbb{Z}_{\geqslant k}$, *we have*

$$\mathrm{gr}_{-k}(\mathcal{S}_{k,Q}(u))=\mathfrak{G}_{kk,K}(u),\quad \frac{(\mathcal{N}-k)!}{(\mathcal{N}-l)!(l-k)!}\mathrm{gr}_{-k}(\mathcal{S}_{k,Q}(u))=\mathfrak{G}_{lk,K}(u). \tag{6.7}$$

In particular, we have

$$\frac{(\mathcal{N}-k)!}{(\mathcal{N}-l)!(l-k)!}\mathfrak{G}_{kk,K}(u)=\mathfrak{G}_{lk,K}(u),$$

$$\mathfrak{D}_{l,K}(u,\partial_u)=\frac{1}{(\mathcal{N}-l)!}\sum_{k=0}^{l}(-1)^{k}\frac{(\mathcal{N}-k)!}{(l-k)!}\mathfrak{G}_{kk,K}(u)\partial_u^{l-k}.$$

Proof. Setting $y=e^{\partial_u}-1$ in (6.6), by Corollary 6.3, we have

$$\sum_{k=0}^{l}(-1)^{k}\frac{(\mathcal{N}-k)!}{(l-k)!}\mathcal{S}_{k,Q}(u)(e^{\partial_u}-1)^{l-k}=(\mathcal{N}-l)!\mathcal{D}_{l,Q}(u,\partial_u)e^{l\partial_u}. \tag{6.8}$$

Then one has

$$(\mathcal{N}-k)!\mathcal{S}_{k,Q}(u)=\sum_{i=0}^{k}(-1)^{i}\frac{(\mathcal{N}-i)!}{(k-i)!}\mathcal{D}_{i,Q}(u,\partial_u)e^{i\partial_u}(e^{\partial_u}-1)^{k-i} \tag{6.9}$$

by the standard identity

$$\sum_{r=0}^{s}\frac{(-1)^r}{r!(s-r)!}=0,\quad s\geqslant 1.$$

Note that $\deg(T(u)-1)=\deg(e^{\partial_u}-1)=-1$ and $\deg(Q-1)\leqslant -1$, we conclude from (6.1) that $\deg(\mathcal{D}_{i,Q}(u,\partial_u))=-i$. Therefore, $\deg(\mathcal{S}_{k,Q}(u))=-k$ by (6.9).

Using $\mathrm{gr}_{-1}(e^{\partial_u}-1)=\partial_u$ and $\mathrm{gr}_{-1}(T(u)-1)=L(u)$, see (6.5), and computing $\mathrm{gr}_{-l}(\mathcal{D}_{l,Q}(u,\partial_u))$ by (6.8) and (6.1), we obtain that

$$(\mathcal{N}-l)!\mathfrak{D}_{l,K}(u,\partial_u)=\sum_{k=0}^{l}(-1)^k\frac{(\mathcal{N}-k)!}{(l-k)!}\mathrm{gr}_{-k}(\mathcal{S}_{k,Q}(u))\partial_u^{l-k},$$

where we also used (6.2). Now (6.7) follows from (6.3). The rests are now obvious. □

Remark 6.6. Note that the last two formulas can also be proved by using similar methods used in Lemma 6.1 and Corollary 6.3. □

6.3. **Recurrence of Bethe vectors.**

In this section, we define similar maps

$$\psi,\qquad \psi(x_1,\dots,x_r),\qquad \tilde{\psi}(x_1,\dots,x_r)$$

for $\mathrm{U}(\mathfrak{gl}_{m|n}[x])$. We shall use the same notations for the counterparts in Section 4.1.

Define the embedding $\psi:\mathrm{U}(\mathfrak{gl}_{\mathcal{N}-1}[x])\hookrightarrow \mathrm{U}(\mathfrak{gl}_{\mathcal{N}}[x])$ by the rule

$$\psi(L_{ab}^{\langle \mathcal{N}-1\rangle}(u))=L_{a+1,b+1}(u),\quad 1\leqslant a,b\leqslant N-1.$$

Define a map $\psi(x_1,\dots,x_r):\mathrm{U}(\mathfrak{gl}_{\mathcal{N}-1}[x])\to \mathrm{U}(\mathfrak{gl}_{\mathcal{N}}[x])\otimes \mathrm{End}(\mathcal{W}^{\otimes r})$ by

$$\psi(x_1,\dots,x_r)(L_{ab}^{\langle \mathcal{N}-1\rangle})=L_{a+1,b+1}(u)\otimes 1^{\otimes r}+\sum_{i=1}^{r}1\otimes\frac{1^{\otimes(i-1)}\otimes E_{ba}^{\langle \mathcal{N}-1\rangle}\otimes 1^{\otimes(r-i)}}{u-x_{r+1-i}}.$$

Define a map $\widetilde{\psi}:\mathrm{U}(\mathfrak{gl}_{\mathcal{N}-1}[x])\to \mathrm{U}(\mathfrak{gl}_{\mathcal{N}}[x])\otimes\mathcal{W}^{\otimes r}$ by

$$\widetilde{\psi}(x_1,\dots,x_r)=\widetilde{\psi}(x_1,\dots,x_r)=\psi(x_1,\dots,x_r)(1\otimes \mathbf{w}_1^{\otimes r}).$$

The following lemmas are straightforward.

Lemma 6.7. *We have* $\widetilde{\psi}(x_1,\dots,x_r)(\mathrm{U}(\mathfrak{gl}_{\mathcal{N}-1}[x]\mathfrak{n}_{\pm}^{\langle\mathcal{N}-1\rangle}[x])\subset \mathrm{U}(\mathfrak{gl}_{\mathcal{N}}[x]\mathfrak{n}_{\pm}[x])\otimes\mathcal{W}^{\otimes r}$.

Similarly, define the embedding $\phi:\mathrm{U}(\mathfrak{gl}_{\mathcal{N}-2}[x])\hookrightarrow \mathrm{U}(\mathfrak{gl}_{\mathcal{N}-1}[x])$ by the rule

$$\phi(L_{ab}^{\langle \mathcal{N}-2\rangle}(u))=L_{a+1,b+1}^{\langle \mathcal{N}-1\rangle}(u),\quad 1\leqslant a,b\leqslant N-2.$$

Lemma 6.8. *We have* $\widetilde{\psi}(x_1,\dots,x_r)\circ\phi=(\psi\circ\phi)\otimes \mathbf{w}_1^{\otimes r}$.

Recall that $\bar{\boldsymbol{\xi}}=(\xi^2,\dots,\xi^{N-1})$ and $\bar{\boldsymbol{t}}=(t_1^2,\dots,t_{\xi^2}^2;\dots;t_1^{N-1},\dots,t_{\xi^{N-1}}^{N-1})$. Define $\mathbb{F}_{\boldsymbol{\xi}}(\boldsymbol{t})$ inductively by

$$\mathbb{F}_{\boldsymbol{\xi}}(\boldsymbol{t})=F^{(1)}(t_1^1)\cdots F^{(\xi^1)}(t_{\xi^1}^1)\widetilde{\psi}(t_1^1,\dots,t_{\xi^1}^1)(\mathbb{F}_{\bar{\boldsymbol{\xi}}}^{\langle\mathcal{N}-1\rangle}(\bar{\boldsymbol{t}}))\tag{6.10}$$

where $F(u)=(L_{12}(u),\dots,L_{1N}(u))=\sum_{a=1}^{N-1}E_{1,a+1}\otimes L_{1,a+1}(u)$ and its coefficients are treated as elements in $\mathrm{Hom}(\mathcal{W},\mathbb{C})\otimes \mathrm{U}(\mathfrak{gl}_{m|n}[x])$.

Recall from Section 6.2 the degree function deg defined on $\mathrm{Y}(\mathfrak{gl}_{m|n})[[u^{-1},\partial_i,\zeta]]$ by $\deg u^{-1}=\deg\partial_u=\deg\zeta=-1$. Extend the degree function to rational expressions in $\boldsymbol{t}$ with coefficients in $\mathrm{Y}(\mathfrak{gl}_{m|n})[[u^{-1},\partial_i,\zeta]]$ by setting $\deg t_i^a=1$ and $\deg(t_i^a-t_j^b)^{-1}=-1$ for all possible a,b,i,j. Note that the maps ψ, $\psi(t_1^1,\dots,t_{\xi_1}^1)$, and $\widetilde{\psi}(t_1^1,\dots,t_{\xi_1}^1)$ respect the degree function and the projections to the associated graded superalgebras. For instance, if $X\in \mathrm{Y}(\mathfrak{gl}_{m|n})$ and $\deg X=k$, then $\deg\psi(X)=k$ and $\psi(\mathrm{gr}_k X)=\mathrm{gr}_k(\psi(X))$.

Proposition 6.9. *We have* $\deg(\widetilde{\mathbb{B}}_{\boldsymbol{\xi}}(\boldsymbol{t}))=-|\boldsymbol{\xi}|$ *and* $\mathrm{gr}_{-|\boldsymbol{\xi}|}(\widetilde{\mathbb{B}}_{\boldsymbol{\xi}}(\boldsymbol{t}))=\mathbb{F}_{\boldsymbol{\xi}}(\boldsymbol{t})$.

Proof. The statements follow from Proposition 4.3, the equality (6.5), and the definition of $\mathbb{F}_{\boldsymbol{\xi}}(\boldsymbol{t})$ by induction. □

6.4. **Proof of Theorem 5.2.** We show Theorem 5.2 by taking the classical limits of Corollary 3.6. We start with recalling the objects we would like to compare between XXX spin chains and Gaudin models.

Let $M_1,\dots,M_\ell$ be highest weight $\mathfrak{gl}_{m|n}$-modules with highest weights $\Lambda_1,\dots,\Lambda_\ell$, $\boldsymbol{z}=(z_1,\dots,z_\ell)$ a sequence of complex numbers. Recall the tensor product of evaluation $\mathrm{Y}(\mathfrak{gl}_{m|n})$-modules $M(\boldsymbol{z})=M_1(z_1)\otimes\cdots\otimes M_\ell(z_\ell)$, the rational difference operator $\mathcal{D}_Q(u,\partial_u)$ defined by quantum Berezinian, see (3.4) and (3.5), and the off-shell Bethe vector $\mathbb{B}_{\boldsymbol{\xi}}^{v^+}(\boldsymbol{t};\boldsymbol{z})$, see (3.17). Consider the rational functions

$$\mathfrak{Q}_{\boldsymbol{\xi},Q}^{a,i}(\boldsymbol{t};\boldsymbol{z};\boldsymbol{\Lambda})=\frac{Q_a}{Q_{a+1}}\prod_{j=1}^{\ell}\frac{t_i^a-z_j+\kappa_a\Lambda_j^a}{t_i^a-z_j+\kappa_{a+1}\Lambda_j^{a+1}}\prod_{j=1}^{\xi^{a-1}}\frac{t_i^a-t_j^{a-1}+\kappa_a}{t_i^a-t_j^{a-1}}$$
$$\times\prod_{j=1,j\neq i}^{\xi^a}\frac{t_i^a-t_j^a-\kappa_a}{t_i^a-t_j^a+\kappa_{a+1}}\prod_{j=1}^{\xi^{a+1}}\frac{t_i^a-t_j^{a+1}}{t_i^a-t_j^{a+1}-\kappa_{a+1}},$$

for $1\leqslant a<N$ and $1\leqslant i\leqslant \xi^a$, cf. the Bethe ansatz equation (3.18), the rational functions $\mathfrak{X}_{\boldsymbol{\xi},Q}^{a}(u;\boldsymbol{t};\boldsymbol{z};\boldsymbol{\Lambda})$, see (3.19), and the rational difference operator $\mathcal{D}_Q(u,\partial_u;\boldsymbol{t};\boldsymbol{z};\boldsymbol{\Lambda})$, see (3.20), which encodes the eigenvalues of transfer matrices for XXX spin chains, see Corollary 3.6.

Similarly, we have the corresponding objects for Gaudin models. Recall the tensor product of evaluation $\mathfrak{gl}_{m|n}[x]$-modules $M(\!(\boldsymbol{z})\!)=M_1(\!(z_1)\!)\otimes\cdots\otimes M_\ell(\!(z_\ell)\!)$, the pseudo-differential operator $\mathfrak{D}_K(u,\partial_u)$ defined by quantum Berezinian, see (5.1), and the off-shell Bethe vector $\mathbb{F}_{\boldsymbol{\xi}}^{v^+}(\boldsymbol{t};\boldsymbol{z})$. Consider the rational functions $\mathfrak{K}_{\boldsymbol{\xi},K}^{a,i}(\boldsymbol{t};\boldsymbol{z};\boldsymbol{\Lambda})$, see (5.6), the rational functions $\mathfrak{X}_{\boldsymbol{\xi},Q}^{a}(u;\boldsymbol{t};\boldsymbol{z};\boldsymbol{\Lambda})$, see (5.7), and the pseudo-differential operator $\mathfrak{D}_K(u,\partial_u;\boldsymbol{t};\boldsymbol{z};\boldsymbol{\Lambda})$, see (5.8).

The objects associated to Gaudin models can be obtained from the corresponding objects for the XXX spin chains by taking the following limit.

Note that $M(\boldsymbol{z})$ and $M(\!(\boldsymbol{z})\!)$ share the same space which we denote by M. Then the following operators

$$T_{ab}^M(u;\boldsymbol{z}):=T_{ab}(u)\big|_{M(\boldsymbol{z})},\qquad L_{ab}^M(u;\boldsymbol{z}):=L_{ab}(u)\big|_{M(\!(\boldsymbol{z})\!)},$$
$$\mathcal{D}_Q^M(u,\partial_u;\boldsymbol{z})=\mathcal{D}_Q(u,\partial_u)\big|_{M(\varepsilon^{-1}\boldsymbol{z})},\qquad \mathfrak{D}_Q^M(u,\partial_u;\boldsymbol{z}):=\mathfrak{D}_Q(u,\partial_u)\big|_{M(\!(\boldsymbol{z})\!)}$$

can be regarded as operators on M depending on the corresponding parameters.

Set $\varepsilon^{-1}\boldsymbol{z}=(\varepsilon^{-1}z_1,\dots,\varepsilon^{-1}z_\ell)$ and $\varepsilon^{-1}\boldsymbol{t}=(\varepsilon^{-1}t_1^1,\dots,\varepsilon^{-1}t_{\xi^{N-1}}^{N-1})$.

Proposition 6.10. *Let* $Q=1+\varepsilon K$. *As* $\varepsilon\to 0$, *we have*

$$T_{ab}^M(\varepsilon^{-1}u;\varepsilon^{-1}\boldsymbol{z})=\delta_{ab}+\kappa_b\varepsilon L_{ab}^M(u;\boldsymbol{z})+O(\varepsilon^2),\tag{6.11}$$

$$\mathfrak{Q}_{\boldsymbol{\xi},Q}^{a,i}(\varepsilon^{-1}\boldsymbol{t};\varepsilon^{-1}\boldsymbol{z};\boldsymbol{\Lambda})=1+\varepsilon\mathfrak{K}_{\boldsymbol{\xi},K}^{a,i}(\boldsymbol{t};\boldsymbol{z};\boldsymbol{\Lambda})+O(\varepsilon^2),\tag{6.12}$$

$$\mathcal{D}_Q^M(\varepsilon^{-1}u,\varepsilon\partial_u;\varepsilon^{-1}\boldsymbol{z})=\varepsilon^{m-n}\mathfrak{D}_Q^M(u,\partial_u;\boldsymbol{z})+O(\varepsilon^{m-n+1}), \tag{6.13}$$

$$\mathcal{D}_{l,Q}(\varepsilon^{-1}u,\varepsilon\partial_u;\varepsilon^{-1}\boldsymbol{t};\varepsilon^{-1}\boldsymbol{z};\boldsymbol{\Lambda})=\varepsilon^{l}\mathfrak{D}_{l,K}(u,\partial_u;\boldsymbol{t};\boldsymbol{z};\boldsymbol{\Lambda})+O(\varepsilon^{l+1}), \tag{6.14}$$

$$\mathcal{D}_Q(\varepsilon^{-1}u,\varepsilon\partial_u;\varepsilon^{-1}\boldsymbol{t};\varepsilon^{-1}\boldsymbol{z};\boldsymbol{\Lambda})=\varepsilon^{m-n}\mathfrak{D}_K(u,\partial_u;\boldsymbol{t};\boldsymbol{z};\boldsymbol{\Lambda})+O(\varepsilon^{m-n+1}), \tag{6.15}$$

$$\mathbb{B}_{\boldsymbol{\xi}}^{v^+}(\varepsilon^{-1}\boldsymbol{t};\varepsilon^{-1}\boldsymbol{z})\prod_{i=1}^{\ell}\prod_{a=1}^{N-1}\prod_{j=1}^{\xi^a}\frac{\varepsilon}{t_j^a-z_i}\prod_{a=1}^{N-2}\prod_{i=1}^{\xi^a}\prod_{j=1}^{\xi^{a+1}}\frac{\varepsilon}{t_j^{a+1}-t_i^a}=\varepsilon^{|\boldsymbol{\xi}|}\mathbb{F}_{\boldsymbol{\xi}}^{v^+}(\boldsymbol{t};\boldsymbol{z})+O(\varepsilon^{|\boldsymbol{\xi}|+1}), \tag{6.16}$$

Proof. The equalities (6.11) and (6.12) are straightforward. The formulas (6.13), (6.14), (6.15) are proved similarly as in Theorem 6.5. The formula (6.16) essentially follows from Proposition 6.9. □

6.5. **Correspondence between** $\mathrm{U}(\mathfrak{gl}_{m|n}[x])$ **and** $\mathrm{U}(\mathfrak{gl}_{n|m}[x])$**.** In this section, we discuss the symmetry between Gaudin models for $\mathfrak{gl}_{m|n}$ and $\mathfrak{gl}_{n|m}$, cf. Section 4.2. We shall use similar conventions as in Section 4.2.

We have the following isomorphism

$$\vartheta:\mathrm{U}(\mathfrak{gl}_{m|n}[x])\to\mathrm{U}(\mathfrak{gl}_{n|m}[x]),\quad L_{ab}(u)\mapsto\widetilde{L}_{b'a'}(u)(-1)^{|a'||b'|+|b'|}$$

For each $l\in\mathbb{Z}_{>0}$, consider another formal differential operator,

$$\mathbb{D}_{l,K}(u,\partial_u)=(\mathrm{str}_{\mathcal{V}^{\otimes l}}\otimes\mathrm{id})\Big(\Big(\overrightarrow{\prod_{1\leqslant i\leqslant l}}\big(\partial_u-K^{(i)}-L^{(i,l+1)}(u)\big)\Big)\mathbb{H}^{(1\cdots l)}_{\{l\}}\Big). \tag{6.17}$$

It is known from [MR14, Theorem 2.13] that

$$\big(\mathrm{Ber}(1+w\mathfrak{Z}_K(u,\partial_u))\big)^{-1}=\sum_{k=0}^{\infty}(-1)^k w^k\mathbb{D}_{l,K}(u,\partial_u). \tag{6.18}$$

We have the following analogous results whose proofs are similar to that of the Yangian case. For a diagonal matrix $K=\sum_{a=1}^N K_aE_{aa}\in\mathrm{End}(\mathcal{V})$, set $\mathsf{K}=\sum_{a=1}^N K_aE_{a'a'}\in\mathrm{End}(\widetilde{\mathcal{V}})$.

Lemma 6.11. *We have*

$$\vartheta(\mathfrak{D}_{l,K}(u,\partial_u))=(-1)^l\widetilde{\mathbb{D}}_{l,\mathsf{K}}(u,\partial_u),\quad\vartheta(\mathbb{D}_{l,K}(u,\partial_u))=(-1)^l\widetilde{\mathfrak{D}}_{l,\mathsf{K}}(u,\partial_u).$$

Proof. The lemma follows immediately from our identification of operators on $\mathcal{V}$ and $\widetilde{\mathcal{V}}$, and the fact that supertranspose respects the supertrace. □

Corollary 6.12. *We have* $\vartheta(\mathrm{Ber}(1+w\mathfrak{Z}_K(u,\partial_u)))=\big(\mathrm{Ber}(1+w\widetilde{\mathfrak{Z}}_{\mathsf{K}}(u,\partial_u))\big)^{-1}$.

Corollary 6.13. *We have* $\vartheta(\mathrm{Ber}(\mathfrak{Z}_K(u,\partial_u)))=\big(\mathrm{Ber}(\widetilde{\mathfrak{Z}}_{\mathsf{K}}(u,\partial_u))\big)^{-1}$.

Proof. Recall that $\mathcal{A}_u^{m|n}=\mathrm{U}(\mathfrak{gl}_{m|n}[x])((u^{-1}))$. Let $\mathcal{A}_u^{m|n}[\partial_u]\subset\mathcal{A}_u^{m|n}((\partial_u^{-1}))$ be the subalgebra of differential operators,

$$\mathcal{A}_u^{m|n}[\partial_u]=\Big\{\sum_{i=0}^{r}a_i\partial_u^i,\ r\in\mathbb{Z}_{\geqslant0},\ a_i\in\mathcal{A}_u^{m|n}\Big\}.$$

Define a linear map $\Phi^{m|n}:\mathcal{A}_u^{m|n}((\partial_u^{-1}))\to\mathcal{A}_u^{m|n}[\partial_u]$,

$$\Phi^{m|n}:\sum_{i=-\infty}^{r}a_i\partial_u^i\to\sum_{i=-\infty}^{r}a_i(w^{-1}+\partial_u)^i,$$

where the right hand side is expanded by the rule: $(w^{-1}+\partial_u)^i = \sum_{j=0}^{\infty}\binom{i}{j}\partial_u^j w^{-i+j}$. Then the map $\Phi^{m|n}$ is an injective homomorphism of superalgebras, see [HM20, Lemma 4.1]. It is also clear that $\vartheta\circ\Phi^{m|n}=\Phi^{n|m}\circ\vartheta$. By the proof of [HM20, Proposition 4.4], we have

$$\begin{aligned} w^{m-n}\Phi^{n|m}(\vartheta(\mathrm{Ber}(\mathfrak{Z}_K(u,\partial_u)))) &= \vartheta(w^{m-n}\Phi^{m|n}(\mathrm{Ber}(\mathfrak{Z}_K(u,\partial_u)))) \\ &= \vartheta(\mathrm{Ber}(1+w\mathfrak{Z}_K(u,\partial_u))) \\ &= \big(\mathrm{Ber}(1+w\widetilde{\mathfrak{Z}}_{\mathsf{K}}(u,\partial_u))\big)^{-1} \\ &= \big(w^{n-m}\Phi^{n|m}(\mathrm{Ber}(\widetilde{\mathfrak{Z}}_{\mathsf{K}}(u,\partial_u))\big)^{-1} \\ &= w^{m-n}\Phi^{n|m}((\mathrm{Ber}(\widetilde{\mathfrak{Z}}_{\mathsf{K}}(u,\partial_u)))^{-1}). \end{aligned}$$

Now the claim follows from the injectivity of $\Phi^{n|m}$. □

Lemma 6.14. *The image of $\mathbb{F}_{\boldsymbol{\xi}}(\boldsymbol{t})$ under the isomorphism ϑ equals to $\widetilde{\mathbb{F}}_{\bar{\boldsymbol{\xi}}}(\bar{\boldsymbol{t}})$ up to sign.*

Proof. The statement follows from the fact that ϖ and ϑ preserve the degrees and the equality

$$\mathrm{gr}^{n|m}_{-|\boldsymbol{\xi}|}\circ\varpi=\vartheta\circ\mathrm{gr}^{m|n}_{-|\boldsymbol{\xi}|},$$

where $\mathrm{gr}^{m|n}_s$ and $\mathrm{gr}^{n|m}_s$ are the graded maps for $\mathrm{Y}(\mathfrak{gl}_{m|n})$ and $\mathrm{Y}(\mathfrak{gl}_{n|m})$, respectively. □

References

[BR08] S. Belliard, E. Ragoucy, *The nested Bethe ansatz for 'all' closed spin chains*, J. Phys A: Math. and Theor. **41** (2008), 295202.

[CF08] A. Chervov, G. Falqui, *Manin matrices and Talalaev's formula*, J. Phys. A: Math. Theor. **41** (2008), 194006.

[CLV22] D. Chernyak, S. Leurent, D. Volin, *Completeness of Wronskian Bethe equations for rational $\mathfrak{gl}_{\mathfrak{m}|\mathfrak{n}}$ spin chains*, Commun. Math. Phys. **391** (2022), 969–1045 .

[FFR94] B. Feigin, E. Frenkel, N. Reshetikhin, *Gaudin model, Bethe ansatz and critical level*, Comm. Math. Phys., **166** (1994), no. 1, 27–62.

[FH15] E. Frenkel, D. Hernandez, *Baxter's Relations and Spectra of Quantum Integrable Models*, Duke Math. J. **164** (2015), no. 12, 2407-2460.

[FJMM17] B. Feigin, M. Jimbo, T. Miwa, E. Mukhin, *Finite type modules and Bethe ansatz equations*, Ann. Henri Poincare **18** (2017), no. 8, 2543-2579.

[GJS22] J. Gu, Y. Jiang, M. Sperling, *Rational Q-systems, Higgsing and Mirror Symmetry*, arXiv:2208.10047.

[HLM19] C.-L. Huang, K. Lu, E. Mukhin, *Solutions of $\mathfrak{gl}_{m|n}$ XXX Bethe ansatz equation and rational difference operators*, J. Phys. A: Math. Theor. **52** (2019), 375204, 31pp.

[HLP+18] A. Hutsalyuk, A. Liashyk, S. Pakuliak, E. Ragoucy, N. Slavnov, *Norm of Bethe vectors in models with $\mathfrak{gl}(m|n)$ symmetry*, Nucl. Phys. B **926** (2018), 256–278.

[HM20] C.-L. Huang, E. Mukhin, *The duality of $\mathfrak{gl}_{m|n}$ and $\mathfrak{gl}_k$ Gaudin models*, J. Algebra **548** (2020), 1–24.

[HMVY19] C.-L. Huang, E. Mukhin, B. Vicedo, C. Young, *The solutions of $\mathfrak{gl}_{m|n}$ Bethe ansatz equation and rational pseudodifferential operators*, Sel. Math. New Ser. (2019) **25**:52.

[KR83] P. Kulish, N. Yu. Reshetikhin, *Diagonalization of $\mathrm{GL}(N)$ invariant transfer-matrices and quantum N-wave system (Lee model)*, J. Phys. A **15** (1983), L591–L596.

[Lu22a] K. Lu, *A note on odd reflections of super Yangian and Bethe ansatz*, Lett. Math. Phys. **112** (2022), no. 2, Paper No. 29, 26 pp.

[Lu22b] K. Lu, *Completeness of Bethe ansatz for Gaudin models associated with $\mathfrak{gl}(1|1)$*, Nuclear Phys. B **980** (2022), Paper No. 115790, 23 pp.

[LM21a] K. Lu, E. Mukhin, *On the supersymmetric XXX spin chains associated with* $\mathfrak{gl}_{1|1}$, Commun. Math. Phys. **386** (2021), 711–747.

[LM21b] K. Lu, E. Mukhin, *Jacobi-Trudi identity and Drinfeld functor for super Yangian*, Int. Math. Res. Not. IMRN, **2021** (2021), no. 21, 16749–16808.

[LM21c] K. Lu, E. Mukhin, *Bethe ansatz equations for orthosymplectic Lie superalgebra and self-dual superspaces*, Ann. Henri Poincaré **22** (2021), 4087–4130.

[Mol07] A. Molev, Yangians and Classical Lie Algebras, Math. Surveys Monogr. **143**, Amer. Math. Soc., Providence, RI, 2007.

[Mol22] A. Molev, *Odd reflections in the Yangian associated with* $\mathfrak{gl}(m|n)$, Lett. Math. Phys. **112** (2022), no. 1, Paper No. 8, 15 pp

[MM17] A. Molev, E. Mukhin, *Eigenvalues of Bethe vectors in the Gaudin model*, Theor. Math. Phys. **192** (2017), no. 3, 1258–1281.

[MR14] A. Molev, E. Ragoucy, *The MacMahon Master Theorem for right quantum superalgebras and higher Sugawara operators for* $\widehat{\mathfrak{gl}}(m|n)$, Moscow Math. J., **14** (2014), no. 1, 83–119.

[MTV06] E. Mukhin, V. Tarasov, A. Varchenko, *Bethe eigenvectors of higher transfer matrices*, J. Stat. Mech. Theor. Exp. (2006) P08002.

[MV17] C. Marboe, D. Volin, *Fast analytic solver of rational Bethe equations*, J. Phys. A **50** (2017) 204002.

[MVY15] E. Mukhin, B. Vicedo, C. Young, *Gaudin models for* $\mathfrak{gl}(m|n)$, J. Math. Phys. **56** (2015), no. 5, 051704, 30 pp.

[NO96] M. Nazarov, G. Olshanski, *Bethe subalgebras in twisted Yangians*, Commun. Math. Phys. **178** (1996), 483–506.

[PRS17] S. Pakuliak, E. Ragoucy, N. Slavnov, *Bethe vectors for models based on the super-Yangian* $Y(\mathfrak{gl}(m|n))$, J. Integrable Syst., **2** (2017), pp. 1–31.

[STF79] E. Sklyanin, L. Takhtadzhyan, L. Faddeev, *Quantum inverse problem method. I*, Theor. Math. Phys. **40** (1979), 688.

[TF79] L. Takhtadzhyan, L. Faddeev, *The quantum method of the inverse problem and the Heisenberg X Y Z model*, Russian Math. Surveys **34** (1979), 11.

[Tal06] D. Talalaev, *The quantum Gaudin system*, Funct. Anal. Appl., **40** (2006), pp. 73–77.

[Tsu97] Z. Tsuboi, *Analytic Bethe ansatz and functional equations for Lie superalgebra* $\mathfrak{sl}(r+1|s+1)$, J. Phys. A **30** (1997), no. 22, 7975.

K.L.: Department of Mathematics, University of Virginia,
141 Cabell Drive, Charlottesville, VA 22904, USA
Email address: kang.lu@virginia.edu